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AN ELEMENTARY TREATISE ON ALGEBRA:

TO WHICH ARE ADDED

EXPONENTIAL EQUATIONS AND LOGARITHMS.

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CONTENTS.

ALGEBRA.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

CHAPTER VI.

CHAPTER VII.

CHAPTER VIII.

CHAPTER IX.

EXPONENTIAL EQUATIONS AND LOGARITHMS.

ALGEBRA.

CHAPTER I.

FUNDAMENTAL PROCESSES OF ALGEBRA.

SECTION I.

Definitions and Notation.

1. *Algebra*, according to the usual definition, is that branch of mathematics in which the quantities considered are represented by the letters of the alphabet, and the operations to be performed upon them are indicated by signs. In this sense it would embrace almost the whole science of mathematics, elementary geometry alone being excepted. It is, consequently, subject in common use to some limitations, which will be more easily understood, when we are advanced in the science.

2. The sign $+$ is called *plus* or *more*, or *the positive sign*, and placed between two quantities denotes that they are to be added together.

Thus $3+5$ is 3 *plus* or *more* 5, and denotes the sum of 3 and 5. Likewise $a+b$ is the sum of a and b, or of the quantities which a and b represent.

3. The sign — is called ***minus*** or ***less***, or ***the negative sign***, and placed between two quantities denotes that the quantity which follows it is to be subtracted from the one which precedes it.

Thus $7 - 2$ is 7 *minus* or *less* 2 and denotes the remainder after subtracting 2 from 7. Likewise $a - b$ is the remainder after subtracting b from a.

4. The sign $\times$ is called the sign of multiplication, and placed between two quantities denotes that they are to be multiplied together. A point is often used instead of this sign, or, when the quantities to be multiplied together are represented by letters, the sign may be altogether omitted.

Thus $3 \times 5 \times 7$, or $3 . 5 . 7$ is the continued product of 3, 5, and 7. Likewise $12 \times a \times b$, or $12 . a . b$, or $12\, a\, b$, is the continued product of 12, a, and b.

5. The *factor* of a product is sometimes called its ***coefficient***, and the numerical factor is called the ***numerical coefficient***. When no coefficient is written, the coefficient may be considered to be unity.

Thus, in the expression $15\, a\, b$, 15 is the numerical coefficient of $a\, b$; and, in the expression $x\, y$, 1 may be regarded as the coefficient of $x\, y$.

6. The continued product of a quantity multiplied repeatedly by itself, is called the *power* of the quantity; and the number of times, which the quantity is taken as a factor, is called the *exponent* of the power.

The power is expressed by writing the quantity

once with the exponent to the right of the quantity, and a little above it. When no exponent is written, the exponent may be considered to be unity.

Thus the fifth power of a is written a^5; but when a stands by itself, it may be regarded as a^1.

7. The *root* of a quantity is the quantity which, multiplied a certain number of times by itself, produces the given quantity; and the *index* of the root is the number of times which the root is contained as a factor in the given quantity.

The sign $\sqrt{}$ is called the *radical* sign, and when prefixed to a quantity indicates that its root is to be extracted, the index of the root being placed to the left of the sign and a little above it. The index 2 is generally omitted, and the radical sign, without any index, is regarded as indicating the *second* or *square* root.

Thus, $\sqrt[2]{a}$ or $\sqrt{a}$ is the square root of a,

$\sqrt[3]{a}$ is the third or cube root of a,

$\sqrt[4]{a}$ is the fourth root of a,

$\sqrt[n]{a}$ is the nth root of a.

8. The signs $\div$ and : are called the signs of division, and either of them placed between two quantities denotes that the quantity which precedes it is to be divided by the one which follows it. The process of division is also indicated by placing the dividend over the divisor with a line between them.

Thus, $a \div b$, or $a : b$, or $\frac{a}{b}$ denotes the quotient of a divided by b.

9. The sign $=$ is called *equal to*, and placed between two quantities denotes that they are equal to each other, and the expression in which this sign occurs is called an *equation*.

Thus, the equation $a = b$ denotes that a is equal to b.

10. The sign $>$ is called *greater than*, and the sign $<$ is called *less than;* and the expression in which either of these signs occurs is called an *inequality*.

Thus, the inequality $a > b$ denotes that a is greater than b; and the inequality $a < b$ denotes that a is less than b; the greater quantity being always placed at the opening of the sign.

11. An *algebraic quantity* is any quantity written in algebraic language.

12. An algebraic quantity, in which the letters are not separated by the signs $+$ and $-$, is called a *monomial*, or a quantity composed of a single term, or simply a *term*.

Thus, $3\,a^2$, $-\,10\,a^3\,x$ are monomials.

13. An algebraic expression composed of several terms, connected together by the signs $+$ and $-$, is called a *polynomial*, one of two terms is called a *binomial*, one of three a *trinomial*, &c.

Thus, $a^2 + b$ is a binomial,

$c + x - y$ is a trinomial, &c.

14. The value of a polynomial is evidently not affected by changing the order of its terms.

Thus, $a - b - c + d$ is the same as $a - c - b + d$, or $a + d - b - c$, or $-\,b + d + a - c$, &c.

Degree, Dimension, Vinculum, Bar, Parenthesis, Similar Terms.

15. Each literal factor of a term is called a ***dimension***, and the ***degree*** of a term is the number of its dimensions.

The degree of a term is, therefore, found by taking the sum of the exponents of its literal factors.

Thus, $7x$ is of one dimension, or of the first degree; $5a^2bc$ is of four dimensions, or of the fourth degree, &c.

16. A polynomial is *homogeneous*, when all its terms are of the same degree.

Thus, $3a - 2b + c$ is homogeneous of the first degree, $8a^3b - 16a^2b^2 + b^4$ is homogeneous of the fourth degree.

17. A *vinculum* or *bar* ———, placed over a quantity, or a *parenthesis* ( ) enclosing it, is used to express that all the terms of the quantity are to be considered together.

Thus, $(a + b + c) \times d$ is the product of $a + b + c$ by d, $\sqrt{\overline{x^2 + y^2}}$, or $\sqrt{}(x^2 + y^2)$ is the square root of $x^2 + y^2$.

The bar is sometimes placed *vertically*.

Thus,
$$\begin{array}{r|cr|cr|l} a & x & +5a^2 & x^2 & -3c & x^3 \\ -2b & & +3 & & +4d & \\ +3c & & -2d & & -1 & \end{array}$$
is the same as
$$(a - 2b + 3c)x + (5a^2 + 3 - 2d)x^2 + (-3c + 4d - 1)x^3.$$

18. *Similar terms* are those in which the literal factors are identical.

Thus,	$7ab$ and $-3ab$ are similar terms,
and	$-5a^4b^3$ and $3a^4b^3$ are similar;
but	$2a^4b^3$ and $2a^3b^4$ are not similar.

19. The terms of a polynomial which are preceded by the sign $+$ are called the *positive* terms, and those which are preceded by the sign $-$ are called the *negative* terms.

When the first term is not preceded by any sign it is to be regarded as *positive*.

20. The following rule for reducing polynomials, which contain similar terms, is too obvious to require demonstration.

Find the sum of the similar positive terms by adding their coefficients, and in the same way the sum of the similar negative terms. The difference of these sums preceded by the sign of the greater, may be substituted as a single term for the terms from which it is obtained.

When these sums are equal they cancel each other, and the corresponding terms are to be omitted.

Thus, $a^2 b - 9 a b^2 + 8 a^2 b + 5 c - 3 a^2 b + 8 a b^2 + 2 a^2 b + c + a b^2 - 8 c$ is the same as $8 a^2 b - 2 c$.

21. EXAMPLES.

1. Reduce the polynomial $10 a^4 + 3 a^4 + 6 a^4 - a^4 - 5 a^4$ to its simplest form. *Ans.* $13 a^4$.

2. Reduce the polynomial $5 a^4 b + 3 \sqrt{a b^2 c} - 7 a b + 17 a b + 2 \sqrt{a b^2 c} - 6 a^4 b - 8 \sqrt{a b^2 c} - 10 a b + 9 a^4 b$ to its simplest form. *Ans.* $8 a^4 b - 3 \sqrt{a b^2 c}$.

3. Reduce the polynomial $3 a - 2 a - 7 f + 3 f + 2 a + 4 f - 3 a$ to its simplest form. *Ans.* 0.

SECTION II.

Addition.

22. *Addition* consists in finding the quantity equivalent to the aggregate or *sum* of several different quantities.

23. *Problem. To find the sum of any given quantities.*

Solution. The following solution requires no demonstration.

The quantities to be added are to be written after each other with the proper sign between them, and the polynomial thus obtained can be reduced to its simplest form by art. 20.

24. EXAMPLES.

1. Find the sum of a and a. *Ans.* $2a$.
2. Find the sum of $11x$ and $9x$. *Ans.* $20x$.
3. Find the sum of $11x$ and $-9x$. *Ans.* $2x$.
4. Find the sum of $-11x$ and $9x$. *Ans.* $-2x$.
5. Find the sum of $-11x$ and $-9x$. *Ans.* $-20x$
6. Find the sum of a and $-b$. *Ans.* $a-b$.
7. Find the sum of $-6f$, $9f$, $13f$, and $-8f$. *Ans.* $8f$.
8. Find the sum of $-12b$, $-4b$ and $13b$. *Ans.* $-3b$.
9. Find the sum of $\sqrt{x}+ax-ab$, $ab-\sqrt{x}+xy$, $ax+xy-4ab$, $\sqrt{x}+\sqrt{x}-x$ and $xy+xy+ax$.

Ans. $2\sqrt{x}+3ax-4ab+4xy-x$.

10. Find the sum of

$$\begin{array}{rrrrr}
7x^2 & -6\sqrt{x} & +5x^3z & +3 & -g \\
-x^2 & 3-\sqrt{x} & & -8 & -g \\
-x^2 & +\sqrt{x} & -3x^3z & -1 & +7g \\
-2x^2 & +3\sqrt{x} & +3x^3z & -1 & -g \\
x^2 & +8\sqrt{x} & -5x^3z & +9 & -g \\
\hline
4x^2 & +3\sqrt{x} & & +2 & +5g.
\end{array}$$

Ans. $4x^2+3\sqrt{x}+2+5g.$

SECTION III.

Subtraction.

25. *Subtraction* consists in finding the *difference* between two quantities.

26. *Problem.* *To subtract one quantity from another.*

Solution. Let A denote the aggregate of all the positive terms of the quantity to be subtracted, and B the aggregate of all its negative terms; then $A-B$ is the quantity to be subtracted, and let C denote the quantity from which it is to be taken.

If A alone be taken from C, the remainder $C-A$ is as much too small as the quantity subtracted is too large, that is, as much as A is larger than $A-B$. The required remainder is, consequently, obtained by increasing $C-A$ by the excess of A above $A-B$, that is, by B, and it is thus found to be $C-A+B$.

The same result would be obtained by adding to C the quantity $A-B$, with its signs reversed, so as to make it $-A+B$. Hence,

To subtract one quantity from another, change the signs of the quantity to be subtracted from $+$ to $-$,

and from — to +, and add it with its signs thus reversed to the quantity from which it is to be taken.

27. EXAMPLES.

1. From a take $b+c$. *Ans.* $a-b-c$.
2. From a take $-b$. *Ans.* $a+b$.
3. From $5a$ take $-5a$. *Ans* $10a$.
4. From $7a$ take $12a$. *Ans.* $-5a$.
5. From $-19a$ take $-20a$. *Ans.* a.
6. From 12 take -7. *Ans.* 19.
7. From -2 take 5. *Ans.* -7.
8. From -11 take -20. *Ans.* 9.
9. From $3a-17b-10b+13a-2a$
take $6b-8a-b-2a+3d+9a-5h$.
Ans. $15a-32b-3d+5h$.

10. Reduce $32a+3b-(5a+17b)$ to its simplest form. *Ans.* $27a-14b$.

11. Reduce $a+b-(2a-3b)-(5a+7b)-(-13a+2b)$ to its simplest form. *Ans.* $7a-5b$.

SECTION IV.

Multiplication.

28. *Problem. To find the continued product of several monomials.*

Solution. The required product is indicated by writing the given monomials after each other with the sign of multiplication between them, and thus a monomial is formed, which is the continued product of all the factors of the given

monomials. But, as the order of the factors may be changed at pleasure, the numerical factors may all be united in one product.

Hence the coefficient of the product of given monomials is the product of their coefficients.

The different powers of the same letter may also be brought together, and since, by art. 6, each exponent denotes the number of times which the letter occurs as a factor in the corresponding term, the number of times which it occurs as a factor in the product must be equal to the sum of the exponents.

Hence every letter which is contained in any of the given factors must be written in the product, with an exponent equal to the sum of all its exponents in the different factors.

29. EXAMPLES.

1. Multiply $a\,b$ by $c\,d\,e$. *Ans.* $a\,b\,c\,d\,e$.

2. Find the continued product of $3\,a\,b$, $2\,c\,d$, and $e\,f\,g$. *Ans.* $6\,a\,b\,c\,d\,e\,f\,g$.

3. Multiply a^m by a^n. *Ans.* a^{m+n}.

4. Find the continued product of $5\,a^3$, a^7, $7\,a^5$, and $3\,a^6$. *Ans.* $105\,a^{21}$.

5. Multiply $7\,a^3\,b^2$ by $10\,a\,b^5\,c^2\,d$. *Ans.* $70\,a^4\,b^7\,c^2\,d$.

6. Find the continued product of $5\,a^3\,b^4$, $a^2\,b^8$, and $4\,a\,b^3\,c$. *Ans.* $20\,a^6\,b^{15}\,c$.

7. Find the continued product of $a^m\,b^p\,c^q$, $a^n\,b^r\,c^s$, and $a^{m+n}\,b$. *Ans.* $a^{2m+2n}\,b^{p+r+1}\,c^{q+s}$

30. *Problem. To find the product of two polynomials.*

Solution. Denote the aggregate of all the positive terms of one factor by A and of the other by B, and those of their negative terms respectively by C and D; and, then, the factors are $A - C$ and $B - D$.

Now if $A - C$ is multiplied by B it is taken as many times too often as there are units in D; so that the required product must be the product of $A - C$ by B, diminished by the product of $A - C$ by D; that is,

$$(A - C)(B - D) = (A - C)B - (A - C)D.$$

Again, by similar reasoning, the product of $A - C$ by B, that is, of B by $A - C$, must be

$$(A - C)B = AB - BC,$$

and that of $(A - C)$ by D must be

$$(A - C)D = AD - CD;$$

and, therefore, the required product is, by art. 26,

$$(A - C)(B - D) = AB - BC - AD + CD.$$

The positive terms of this product, AB and CD, are obtained from the product of the positive terms A and B, or from that of the negative terms $-C$ and $-D$; but the negative terms of the product, as $-BC$ and $-AD$, are obtained from the product of the negative term of one factor by the positive term of the other, as $-C$ by B or $-D$ by A. Hence,

The product of two polynomials is obtained by multiplying each term of one factor by each term of the other, as in art. 28, and the product of two terms which have the same sign is to be affected with the sign $+$, while the product of two terms which have contrary signs is to be affected by the sign $-$.

The result is to be reduced as in art. 20.

31. EXAMPLES.

1. Multiply x^2+y^2 by $x+y$.
Ans. $x^3+x^2y+xy^2+y^3$.

2. Multiply x^5+xy^6+7ax by $ax+5ax$.
Ans. $6ax^6+6ax^2y^6+42a^2x^2$.

3. Multiply $-a$ by b. Ans. $-ab$.

4. Multiply a by $-b$. Ans. $-ab$.

5. Multiply $-a$ by $-b$. Ans. ab.

6. Multiply $-3a$ by $14c$. Ans. $-42ac$.

7. Multiply $-6a^3b^2$ by $-11ab^3c$. Ans. $66a^4b^5c$.

8. Find the continued product of $-a$, $-a$, $-a$, and $-a$. Ans. a^4.

9. Find the continued product of $-a^2b$, c^2e, $-a$, $-e^3x^2$, c, $-2ax$, $-3abex$, -7, and b^3x^3.
Ans. $42a^5b^5c^3e^5x^7$.

10. Find the continued product of $7abx$, $-ax$, $-x$, b^2x^7, $-2b$, -3, and $-5a^7b^3x^5$.
Ans. $-210a^9b^7x^{15}$.

11. Multiply $a+b$ by $c+d$.
Ans. $ac+ad+bc+bd$.

12. Multiply a^3+b^2-c by a^2-b^3.
Ans. $a^5-a^3b^3+a^2b^2-a^2c-b^5+b^3c$.

13. Multiply $a+b+c$ by $a+b-c$.
Ans. $a^2+2ab+b^2-c^2$.

14. Multiply x^2-3x-7 by $x-2$.
Ans. x^3-5x^2-x+14.

15. Multiply $a^2+a^4+a^6$ by a^2-1. Ans. a^8-a^2.

16. Multiply $8a^9b^3+36a^8b^4+54a^7b^5+27a^6b^6$ by $8a^9b^3-36a^8b^4+54a^7b^5-27a^6b^6$.
Ans. $64a^{18}b^6-432a^{16}b^8+972a^{14}b^{10}-729a^{12}b^{12}$.

17. Find the continued product of $3x+2y$, $2x-3y$, $-x+y$, and $-2x\cdot\ y$.

Ans. $12x^4-16x^3y-13x^2y^2+11xy^3+6y^4$.

18. Multiply $a+b$ by $a-b$. *Ans.* a^2-b^2.

19. Multiply $2a^3x+7a^2x^5$ by $2a^3x-7a^2x^5$.

Ans. $4a^6x^2-49a^4x^{10}$.

32. *Corollary.* The continued product of several monomials is, as in examples 8 and 9, positive, when the number of negative factors is even; and it is negative, as in example 10, when the number of negative factors is odd.

33. *Corollary.* The product of the sum of two numbers by their difference is, as in examples 18 and 19, equal to the difference of their squares.

34. *Theorem.* The product of homogeneous polynomials is also homogeneous, and the degree of the product is equal to the sum of the degrees of the factors.

Demonstration. For the number of factors in each term of the product is equal to the sum of the numbers of factors in all the terms from which it is obtained; and, therefore, by art. 15, the degree of each term of the product is equal to the sum of the degrees of the factors. Thus, in example 16, the degree of each factor is 12, and that of the product is $12+12$ or 24.

SECTION V.

Division

35. ***Problem.*** ***To divide one monomial by another.***

Solution. Since the dividend is the product of the divisor and quotient, the quotient must be obtained by suppressing in the dividend all the factors of the divisor which are explicitly contained in the dividend, and simply indicating the division with regard to the remaining factors of the divisor. Hence, from art. 28,

Suppress the greatest common factor of the numerical coefficients.

Suppress each letter of the divisor or dividend in the term in which it has the least exponent, and retain it in the other term, giving it an exponent equal to the difference of its exponents in the two terms. But when a letter occurs in only one term, it is to be retained in that term, with its exponent unchanged.

The required quotient is, then, equal to the quotient of the remaining portion of the dividend divided by that of the divisor, and may be indicated as in art. 8; or, when the divisor is reduced to unity, the quotient is simply equal to the remaining portion of the dividend.

The sign of the quotient must, from art. 30, be the same as that of the divisor when the dividend is positive, and it must be the reverse of that of the divisor when the dividend is negative; whence we readily obtain the rule.

When the divisor and dividend are both affected by the same sign, the quotient is positive; but when they are affected by contrary signs, the quotient is negative.

The rule for the signs in both division and multiplication may be expressed still more concisely as follows.

Like signs give $+$; *unlike signs give* $-$.

36. EXAMPLES.

1. Divide $65\,a\,b$ by $5\,a$. *Ans.* $\frac{13\,b}{1} = 13\,b$.
2. Divide $-132\,a^5\,b^3\,c$ by $11\,a^3\,b^3$. *Ans.* $-12\,a^2\,c$.
3. Divide $144\,a\,b^3\,c^2\,d^8\,e$ by $-112\,a\,b^4\,c\,e^7\,h$. *Ans.* $-\frac{9\,c\,d^8}{7\,b\,e^6\,h}$.
4. Divide -135 by $-5\,a$. *Ans.* $\frac{27}{a}$.
5. Divide $7\,a^3\,x^2$ by $21\,a^5\,x^2$. *Ans.* $\frac{1}{3\,a^2}$.
6. Divide a^m by a^n. *Ans.* a^{m-n}.
7. Divide $-3\,a^m\,b^n$ by $-4\,a^p\,b^q\,c^r$. *Ans.* $\frac{3\,a^{m-p}\,b^{n-q}}{4\,c^r}$.
8. Divide a by $-a$. *Ans.* -1.
9. Divide $-a$ by a. *Ans.* -1.
10. Divide $-a$ by $-a$. *Ans.* 1.

37. *Corollary.* If the rule for the exponents is applied to the case in which the exponent of a letter in the dividend is equal to its exponent in the divisor, when, for instance, a^m is to be divided by a^m, the exponent of the letter in the

quotient becomes zero. But the quotient of a quantity divided by itself is unity.

Whence any quantity with an exponent equal to zero is unity.

Thus, $a^m \div a^m = a^0 = 1.$

38. *Corollary.* When, in example 6 of art. 30, the exponent n of a in the divisor is greater than its exponent m in the dividend, the exponent $m - n$ in the quotient is negative; and a negative exponent is thus substituted for the usual fractional form of the quotient.

Thus, if m is zero, we have

$$a^0 \div a^n = 1 \div a^n = \frac{1}{a^n} = a^{-n}.$$

In the same way we should have

$$a\,b^2\,c^3 \div a^5\,b^2\,c^8\,d = a^1\,b^2\,c^3 \div a^5\,b^2\,c^8\,d^1 = a^{-4}\,c^{-5}\,d^{-1}.$$

Any quotient of monomials may thus be expressed by means of negative exponents without using fractional forms.

39. EXAMPLES.

1. Divide $5\,a^4\,b^3\,c^2\,d$ by $15\,a\,b^5\,c^2\,d^3\,e^2$.

   Ans. $3^{-1}\,a^3\,b^{-2}\,d^{-2}\,e^{-2}$

2. Divide $6\,a^7\,b$ by $9\,a\,b^7$.

   Ans. $\frac{2}{3}\,a^6\,b^{-6} = 2\,.\,3^{-1}\,a^6\,b^{-6}.$

3. Divide 1 by $8\,a^{11}\,b$.

   Ans. $\frac{1}{8}\,a^{-11}\,b^{-1} = 8^{-1}\,a^{-11}\,b^{-1}.$

4. Divide 3 by a. *Ans.* $3\,a^{-1}.$

40. *Corollary.* Quantities, thus expressed by means of fractional exponents, may be used in all

calculations, and may be added, subtracted, multiplied, or divided by the rules already given, the signs being carefully attended to.

41. EXAMPLES.

1. Find the sum of $7a^{-3}+9a^m b^{-p}-6ab^{-2}c^4$, $-3a^{-3}$, $5a^m b^{-p}+11ab^{-2}c^4$, $a^{-3}-14a^m b^{-p}$.
Ans. $5a^{-3}+5ab^{-2}c^4$.

2. Reduce the polynomial $9a^{-3}b^{-2}c^4-7ba^{-3}+(18a^{-3}b-5a^n b^m+c^x-3.2^5)-(3a^n b^m-a^{-3}b^{-2}c^4+3c^x-5.2^5)$ to its simplest form.
Ans. $10a^{-3}b^{-2}c^4+11a^{-3}b-8a^n b^m-2c^x+2.2^5$.

3. Multiply a^{-m} by a^n. *Ans.* $a^{-m+n}=a^{n-m}$.

4. Multiply a^m by a^{-n}. *Ans.* a^{m-n}.

5. Multiply a^{-m} by a^{-n}. *Ans.* $a^{-m-n}=a^{-(m+n)}$.

6. Find the continued product of $11a^{-2}$, $-2a^{-5}$, $4a^6$, and $-9a^7$. *Ans.* $792a^6$.

7. Find the continued product of $2a^{-3}$, $7a^{-9}$, and $-3a^6$. *Ans.* $-42a^{-6}=-\frac{42}{a^6}$.

8. Find the continued product of $5a^3b^{-4}$, $10a^2b^5c$, and $-3a^7$. *Ans.* $-150a^{12}bc$.

9. Multiply $-13a^{-1}bc^{-3}$ by $-4a^{-3}b^{-6}c^2$.
Ans. $52a^{-4}b^{-5}c^{-1}$.

10. Divide a^{-m} by a^n. *Ans.* $a^{-m-n}=a^{-(m+n)}$.

11. Divide a^m by $^{-n}$. *Ans.* a^{m+n}.

12. Divide a^{-m} by a^{-n}. *Ans.* $a^{-m+n}=a^{n-m}$.

13. Divide $14a^{-8}bc^2d^{-1}e$ by $2ab^{-3}c^5dh$.
Ans. $7a^{-9}b^4c^{-2}d^{-2}eh^{-1}$

14. Divide $-3a^m$ by $2a^{m+n}bc^{-1}$.
Ans. $-\frac{3}{2}a^{-n}b^{-1}c$.

42. ***Problem.*** ***To divide one polynomial by another.***

Solution. The term of the dividend, which contains the highest power of any letter, must be the product of the term of the divisor which contains the highest power of the same letter, multiplied by the term of the quotient which contains the highest power of the same letter.

A term of the quotient is consequently obtained by dividing, as in art. 35, the term of the dividend which contains the highest power of any letter by that term of the divisor which contains the highest power of the same letter.

But the dividend is the sum of the products of the divisor by all the terms of the quotient; and, therefore,

If the product of the divisor by the term just found is subtracted from the dividend, the remainder must be equal to the sum of the products of the divisor by the remaining terms of the quotient, and may be used as a new dividend to obtain another term of the quotient.

By pursuing this process until the dividend is entirely exhausted, all the terms of the quotient may be obtained.

It facilitates the application of this method to arrange the terms of the dividend and divisor according to the powers of some letter, the term which contains the highest power being placed first, that which contains the next to the highest power being placed next, and so on.

43. EXAMPLES.

1. Divide $-16\,a^3\,x^3 + a^6 + 64\,x^6$ by $4\,x^2 + a^2 - 4\,a\,x$.

Solution. In the following solution the dividend and divisor are arranged according to the powers of the letter x; the divisor is placed at the right of the dividend with the quotient below it.

As each term of the quotient is obtained, its product by the divisor is placed below the dividend or remainder from which it is obtained, and is subtracted from this dividend or remainder.

$$
\begin{array}{l|l}
64\,x^6 - 16\,a^3\,x^3 + a^6 & 4\,x^2 - 4\,a\,x + a^2 = \text{Divisor.} \\
\underline{64\,x^6 - 64\,a\,x^5 + 16\,a^2\,x^4} & \overline{16\,x^4 + 16\,a\,x^3 + 12\,a^2\,x^2 + 4\,a^3\,x + a^4} \\
64\,a\,x^5 - 16\,a^2\,x^4 - 16\,a^3\,x^3 + a^6 = \text{1st Remainder.} & \\
\underline{64\,a\,x^5 - 64\,a^2\,x^4 + 16\,a^3\,x^3} & \\
48\,a^2\,x^4 - 32\,a^3\,x^3 + a^6 = \text{2d Remainder.} & \\
\underline{48\,a^2\,x^4 - 48\,a^3\,x^3 + 12\,a^4\,x^2} & \\
16\,a^3\,x^3 - 12\,a^4\,x^2 + a^6 = \text{3d Remainder.} & \\
\underline{16\,a^3\,x^3 - 16\,a^4\,x^2 + 4\,a^5\,x} & \\
4\,a^4\,x^2 - 4\,a^5\,x + a^6 = \text{4th Remainder.} & \\
\underline{4\,a^4\,x^2 - 4\,a^5\,x + a^6} & \\
0. &
\end{array}
$$

Ans. $16\,x^4 + 16\,a\,x^3 + 12\,a^2\,x^2 + 4\,a^3\,x + a^4$

2. Divide $b\,c^3 - c^3\,x$ by c^3. *Ans* $b - x$.

3. Divide $a^2 + 2\,a\,b + b^2$ by $a + b$. *Ans.* $a + b$.

4. Divide $-a^8\,b^4 + 15\,a^{11}\,b^5 - 48\,a^{14}\,b^6 - 20\,a^{17}\,b^7$ by $10\,a^9\,b^2 - a^6\,b$. *Ans.* $a^2\,b^3 - 5\,a^5\,b^4 - 2\,a^8\,b^5$.

5. Divide $1 - 18\,z^2 + 81\,z^4$ by $1 + 6\,z + 9\,z^2$.
Ans. $1 - 6\,z + 9\,z^2$.

6. Divide $81\,a^8 + 16\,b^{12} - 72\,a^4\,b^6$ by $9\,a^4 + 12\,a^2\,b^3 + 4\,b^6$. *Ans.* $9\,a^4 - 12\,a^2\,b^3 + 4\,b^6$.

7. Divide $x^{6m} - 3\,x^{4m}\,y^{2n} + 3\,x^{2m}\,y^{4n} - y^{6n}$ by $x^{3m} - 3\,x^{2m}\,y^{n} + 3\,x^{m}\,y^{2n} - y^{3n}$.

Ans. $x^{3m} + 3\,x^{2m}\,y^{n} + 3\,x^{m}\,y^{2n} + y^{3n}$.

8. Divide $-1 + a^3\,n^3$ by $-1 + a\,n$.

Ans. $1 + a\,n + a^2\,n^2$.

9. Divide $2\,a^4 - 13\,a^3\,b + 31\,a^2\,b^2 - 38\,a\,b^3 + 24\,b^4$ by $2\,a^2 - 3\,a\,b + 4\,b^2$. *Ans.* $a^2 - 5\,a\,b + 6\,b^2$.

10. Divide $a^2 - b^2$ by $a - b$. *Ans.* $a + b$.

11. Divide $a^3 - b^3$ by $a - b$. *Ans.* $a^2 + a\,b + b^2$.

12. Divide $a^4 - b^4$ by $a - b$.

Ans. $a^3 + a^2\,b + a\,b^2 + b^3$.

13. Divide $a^5 - b^5$ by $a - b$.

Ans. $a^4 + a^3\,b + a^2\,b^2 + a\,b^3 + b^4$.

44. *Corollary. The quotient can be obtained with equal facility by using the terms which contain the lowest powers of a letter instead of those which contain the highest powers.*

In this case, it is more convenient to place the term containing the lowest power first, and that containing the next lowest next, and so on.

This order of terms is called *an arrangement according to the ascending powers of the letter;* whereas that of the preceding article is called *an arrangement according to the descending powers of the letter.*

45. *Corollary.* Negative powers are considered to be lower than positive powers, or than the power zero, and the larger the *absolute value* of the exponent the lower the power.

Thus $a^5\,x^{-6} - a^4\,x^{-3} + a^3 + a^{-1}\,x + a^{-2}\,x^2$,

is arranged according to the ascending powers of x, and according to the descending powers of a.

46. EXAMPLES.

1. Divide $a^4 + a^2 - a^{-2} - a^{-4}$ by $a^2 - a^{-2}$.
Ans. $a^2 + 1 + a^{-2}$.

2. Divide $4\,a^4 b^{-6} + 12\,a^3 b^{-5} + 9\,a^2 b^{-4} - b^{-2} + 2\,a^{-2} - a^{-4} b^2$ by $2\,a^2 b^{-3} + 3\,a\,b^{-2} - b^{-1} + a^{-2} b$.
Ans. $2\,a^2 b^{-3} + 3\,a\,b^{-2} + b^{-1} - a^{-2} b$.

47. In the course of algebraic investigations, it is often convenient to separate a quantity into its factors. This is done, when one of the factors is known, by dividing by the known factor, and the quotient is the other factor.

And when a letter occurs as a factor of all the terms of a quantity, it is a factor of the quantity, and may be taken out as a factor, with an exponent equal to the lowest exponent which it has in any term, and indeed by means of negative exponents any monomial may be taken out as a factor of a quantity.

48. EXAMPLES.

1. Take out $3\,a^2 b$ as a factor of $15\,a^5 b^2 + 6\,a^3 b + 9\,a^2 b^2 + 3\,a^2 b$. *Ans.* $3\,a^2 b\,(5\,a^3 b + 2\,a + 3\,b + 1)$.

2. Take out a^m as a factor of $3\,a^{m+1} + 2\,a^m$.
Ans. $a^m\,(3\,a + 2)$.

3. Take out $2\,a^3 b^5 c$ as a factor of $6\,a^6 b^7 c^2 + 6\,a\,b^8 c - 2\,a\,b + 2 - a^2 c$.
Ans. $2\,a^3 b^5 c\,(3\,a^3 b^2 c + 3\,a^{-2} b^3 - a^{-2} b^{-4} c^{-1} + a^{-3} b^{-5} c^{-1} - 2^{-1} a^{-1} b^{-5}$.

4. Take out b as a factor of $a^{n-1} b - b^n$.
Ans. $b\,(a^{n-1} - b^{n-1})$.

49. *Theorem.* The difference of two integral positive powers of the same degree is divisible by the difference of their roots.

Thus, $a^n - b^n$ is divisible by $a - b$.

Demonstration. Divide $a^n - b^n$ by $a - b$, as in art. 42, proceeding only to the first remainder, as follows.

$$\begin{array}{r|l} a^n - b^n & a - b \\ \cline{2-2} a^n - a^{n-1}\,b & a^{n-1} \\ \cline{1-1} \end{array}$$

1st Remainder $= a^{n-1}\,b - b^n = b\,(a^{n-1} - b^{n-1})$.

Now, if the factor $a^{n-1} - b^{n-1}$ of this remainder is divisible by $a - b$, the remainder itself is divisible by $a - b$, and therefore $a^n - b^n$ is also divisible by $a - b$; that is, if the proposition is true for any power, as the $(n-1)$st, it also holds for the nth, or the next greater.

But from examples, 10, 11, 12, 13 of art. 43, the proposition holds for the 2d, 3d, 4th, and 5th; and therefore it must be true for the 6th, 7th, 8th, &c. powers; that is, for any positive integral power.

50. *Corollary.* The division of $a^n - b^n$ by $a - b$ may be continued for the purpose of showing the form of the quotient,

$$\begin{array}{l|l} a^n - b^n & a - b \\ \cline{2-2} a^n - a^{n-1}\,b & a^{n-1} + a^{n-2}\,b + a^{n-3}\,b^2 + \text{\&c.}\ldots\ldots + a\,b^{n-2} + b^{n-1} \\ \cline{1-1} \quad a^{n-1}\,b - b^n & \\ \quad a^{n-1}\,b - a^{n-2}\,b^2 & \\ \cline{1-1} \qquad a^{n-2}\,b^2 - b^n & \\ \qquad a^{n-2}\,b^2 - a^{n-3}\,b^3 & \\ \cline{1-1} \qquad\quad \text{\&c.}\ \ldots & \\ \qquad\quad \ldots\ldots & \\ \cline{1-1} \qquad\qquad a^2\,b^{n-2} - b^n & \\ \qquad\qquad a^2\,b^{n-2} - a\,b^{n-1} & \\ \cline{1-1} \qquad\qquad\quad a\,b^{n-1} - b^n & \\ \qquad\qquad\quad a\,b^{n-1} - b^n & \\ \cline{1-1} \end{array}$$

that is,

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \&c.\ldots + ab^{n-2} + b^{n-1},$$

so that each term of the quotient is obtained from the preceding term by diminishing the exponent of a by unity and increasing that of b by unity; and the number of terms is equal to the exponent n.

51. *Corollary.* If b is put equal to a in the preceding quotient, each of its terms becomes equal to a^{n-1}, which gives the peculiar result

$$\frac{a^n - a^n}{a - a} = n\,a^{n-1}$$

52. There are sometimes two or more terms in the divisor, or in the dividend, or in both, which contain the same highest power of the letter according to which the terms are arranged.

In this case, these terms are to be united in one by taking out their common factor; and the compound terms thus formed are to be used as simple ones. It is more convenient to arrange the terms which contain the same power of the letter in a column under each other, the vertical bar being used as in art. 17*; and to arrange the terms in the vertical columns according to the powers of some letter common to them.*

53. EXAMPLES.

1. Divide $a^2x^3 - b^2x^3 - 4abx^2 - 2a^2x + 2abx + a^2 - b^2$ by $ax - bx - a - b$.

Division of Polynomials.

Solution.

$$
\begin{array}{r|l|l|l}
\begin{matrix} a^2 \\ -b^2 \end{matrix} & \begin{matrix} x^3 - 4abx^2 - 2a^2 \\ \qquad\qquad +2ab \end{matrix} & \begin{matrix} x + a^2 \\ \quad -b^2 \end{matrix} \\
\begin{matrix} a^2 \\ -b^2 \end{matrix} & \begin{matrix} x^3 - a^2 \\ \quad -2ab \\ \quad -b^2 \end{matrix} & x^2 \\
\hline
\text{1st Rem.}\ \begin{matrix} a^2 \\ -2ab \\ +b^2 \end{matrix} & \begin{matrix} x^2 - 2a^2 \\ \quad +2ab \end{matrix} & \begin{matrix} x + a^2 \\ \quad -b^2 \end{matrix} \\
\begin{matrix} a^2 \\ -2ab \\ +b^2 \end{matrix} & \begin{matrix} x^2 - a^2 \\ \quad +b^2 \end{matrix} & x \\
\hline
\text{2d Remainder} & \begin{matrix} -a^2 \\ +2ab \\ -b^2 \end{matrix} & \begin{matrix} x + a^2 \\ \quad -b^2 \end{matrix} \\
 & \begin{matrix} -a^2 \\ +2ab \\ -b^2 \end{matrix} & \begin{matrix} x + a^2 \\ \quad -b^2 \end{matrix} \\
\hline
\text{3d Remainder} & 0. &
\end{array}
\qquad
\begin{array}{r|l|l}
\begin{matrix} a \\ -b \end{matrix} & \begin{matrix} x - a \\ \quad -b \end{matrix} & \\
\hline
\begin{matrix} a \\ +b \end{matrix} & \begin{matrix} x^2 + a \\ \quad -b \end{matrix} & \begin{matrix} x - a \\ \quad +b \end{matrix}
\end{array}
$$

In this quotient, the coefficient $a + b$ of x^2, the coefficient $a - b$ of x and the term $-a + b$ are successively obtained by dividing the coefficient $a^2 - b^2$ of x^3 in the dividend, the coefficient $a^2 - 2ab + b^2$ of x^2 in the first remainder, and the coefficient $-a^2 + 2ab - b^2$ of x in the second remainder, by the coefficient $a - b$ of x in the divisor.

Ans. $(a + b)x^2 + (a - b)x - (a - b)$.

2. Divide $(6b - 10)a^4 - (7b^2 - 23b + 20)a^3 - (3b^3 - 22b^2 + 31b - 5)a^2 + (4b^3 - 9b^2 + 5b - 5)a + b^2 - 2b$ by $(3b - 5)a + b^2 - 2b$.

Ans. $2a^3 - (3b - 4)a^2 + (4b - 1)a + 1$.

3. Divide $-a^6 - (b^2 - 2c^2)a^4 + (b^4 - c^4)a^2 + (b^6 + 2b^4c^2 + b^2c^4)$ by $a^2 - b^2 - c^2$.

Ans. $-a^4 - (2b^2 - c^2)a^2 - b^4 - b^2c^2$.

Division of Polynomials.

4. Divide

$$\begin{array}{r|r|r|r|l} y^3 & x^5 - 3\,y^4 & x^4 - y^5 & x^3 + 3\,y^6 & x^2 \\ -\,y & -\,3\,y^3 & +\,10\,y^4 & -\,3\,y^5 & \\ & +\,3\,y^2 & +\,3\,y^3 & -\,9\,y^4 & \\ & +\,3\,y & -\,10\,y^2 & +\,3\,y^3 & \\ & & -\,2\,y & +\,6\,y^2 & \end{array}$$

by

$$\begin{array}{r|r|r|} y & x^2 - 3\,y & x - y^3 \\ +\,1 & -\,3 & +\,3\,y \\ & & +\,2 \end{array}$$

Ans. $\begin{array}{r|r|l} y^2 & x^3 - 3\,y^3 & x^2. \\ -\,y & +\,3\,y^2 & \end{array}$

CHAPTER II.

FRACTIONS AND PROPORTIONS.

SECTION I.

Reduction of Fractions.

54. When a quotient is expressed by placing the dividend over the divisor with a line between them, it is called a *fraction;* its dividend is called the *numerator* of the fraction, and its divisor the *denominator* of the fraction; and the numerator and denominator of a fraction are called the *terms* of the fraction.

When a quotient is expressed by the sign (:) it is called a *ratio;* its dividend is called the *antecedent* of the ratio, and its divisor the *consequent* of the ratio; and the antecedent and consequent of a ratio are called the *terms* of the ratio.

55. *Theorem. The value of a fraction, or of a ratio, is not changed by multiplying or dividing both its terms by the same quantity.*

Proof. For dividing both these terms by a quantity is the same as striking out a factor common to the two terms of a quotient, which, as is evident from art. 35, does not affect the value of the quotient. Also multiplying both terms by a quantity is only the reverse of the preceding process, and cannot therefore change the value of the fraction or ratio.

56. The terms of a fraction can often be simplified

by dividing them by a common factor or divisor. But when they have no common divisor, the fraction is said to be *in its lowest terms.*

A fraction is, consequently, reduced to its lowest terms, by dividing its terms by their greatest common factor or divisor.

57. *Problem.* *To find the greatest common divisor of several monomials.*

Solution. It is equal to the product of the greatest common divisor of the coefficients, by those different literal factors which are common to all the monomials, each literal factor being raised to the lowest power which it has in either of the monomials.

58. EXAMPLES.

1. Find the greatest common divisor of $75\,a^3\,b^8\,c\,d^{11}\,x^9$ and $50\,a^3\,c^2\,d^{11}\,x^5$. *Ans.* $25\,a^3\,c\,d^{11}\,x^5$.

2. Reduce the fraction $\frac{121\,a\,b^2\,c^3\,d^4\,x^5\,y^6}{132\,a^6\,b^5\,c^4\,d^3\,x^2\,y}$ to its lowest terms. *Ans.* $\frac{11\,d\,x^3\,y^5}{12\,a^5\,b^3\,c}$.

3. Reduce the fraction $\frac{17\,a^3\,b}{51\,a\,b^5}$ to its lowest terms. *Ans.* $\frac{a^2}{3\,b^4}$.

59. *Lemma.* The greatest common divisor of two quantities is the same with the greatest common divisor of the least of them, and of their remainder after division.

Demonstration. Let the greatest of the two quantities be A, and the least B; let the entire part of their quotient after division be Q, and the remainder R; and let the greatest

common divisor of A and B be D, and that of B and R be E. We are to prove that

$$D = E.$$

Now since R is the remainder of the division of A by B, we have

$$R = A - B \,.\, Q;$$

and, consequently, D, which is a divisor of A and B, must divide R; that is, D is a common divisor of B and R, and cannot therefore be greater than their greatest common divisor E.

Again, we have

$$A = R + B \,.\, Q,$$

and, consequently, E, which is a divisor of B and R, must divide A; that is, E is a common divisor of A and B, and cannot therefore be greater than their greatest common divisor D.

D and E, then, are two quantities such that neither is greater than the other; and must therefore be equal.

60. *Problem. To find the greatest common divisor of any two quantities.*

Solution. Divide the greater quantity by the less, and the remainder, which is less than either of the given quantities, is, by the preceding article, divisible by the greatest common divisor.

In the same way, from this remainder and the divisor a still smaller remainder can be found, which is divisible by the greatest common divisor; and, by continuing this process with each remainder and its corresponding divisor, quantities smaller and smaller are found, which are all divisible by the greatest common divisor, until at length the common divisor itself must be attained.

Greatest Common Divisor.

The greatest common divisor, when obtained, is at once recognised from the fact, that the preceding divisor is exactly divisible by it without any remainder.

The quantity thus obtained, must be the greatest common divisor required; for, from the preceding article, the greatest common divisor of each remainder and its divisor is the same with that of the divisor and its dividend, that is, of the preceding remainder and its divisor; hence, it is the same with that of any divisor and its dividend, or with that of the given quantities.

61. *Corollary.* ***When the remainders decrease to unity, the given quantities have no common divisor, and are said to be incommensurable or prime to each other.***

62. EXAMPLES.

1. Find the greatest common divisor of 1825 and 1995

Solution.

$$\begin{array}{r|l}
1995 & 1825 \\
1825 & 1 \\
\hline
1825 & 170, \text{ 1st Rem.} \\
1700 & 10 \\
\hline
170 & 125, \text{ 2d Rem.} \\
125 & 1 \\
\hline
125 & 45, \text{ 3d Rem.} \\
90 & 2 \\
\hline
45 & 35, \text{ 4th Rem.} \\
35 & 1 \\
\hline
35 & 10, \text{ 5th Rem.} \\
30 & 3 \\
\hline
10 & 5, \text{ 6th Rem.} \\
10 & 2
\end{array}$$

Ans. 5

This process may be written more neatly and concisely as follows.

1995	1825	1
1825	1700	10
170	125	1
125	90	2
45	35	1
35	30	3
10	5	2
10		

2. Find the greatest common divisor of 13212 and 1851.
Ans. 3.

3. Find the greatest common divisor of 1221 and 333.
Ans. 111.

63. The above rule requires some modification in its application to polynomials.

Thus it frequently happens in the successive divisions, that the term of the dividend, from which the term of the quotient is to be obtained, is not divisible by the corresponding term of the divisor. This, sometimes, arises from a monomial factor of the divisor which is prime to the dividend, and which may be suppressed.

For, since the greatest common divisor of two quantities is only the product of their common factors, it is not affected by any factor of the one quantity which is prime to the other.

Hence any monomial factor of either dividend or its divisor is to be suppressed which is prime to the other of these two quantities, and when there is such a factor it is readily obtained by inspection.

But if, after this reduction, the first term of the dividend, when arranged according to the powers of some letter, is still not divisible by the first term of the divisor similarly arranged; it follows from the preceding reasoning that it can lead to no error to

Multiply the dividend by some monomial factor which will render its first term divisible by the first term of the divisor, and which is prime to the reduced divisor. Such a factor can always be obtained by simple inspection.

When the given quantities have any common monomial factor it is easily obtained from inspection, and it should be suppressed at first, and afterwards multiplied by the greatest common divisor of the remaining polynomials.

Since any quantity which is divisible by A is also divisible by $-A$; and any quantity which is divisible by $-A$ is also divisible by A;

All the signs of any divisor may be reversed at pleasure.

64. EXAMPLES.

1. Find the greatest common divisor of $6\,a^2\,x^3 + 21\,a^3\,x^2 - 27\,a^5$ and $4\,x^4 + 5\,a^2\,x^2 + 21\,a^3\,x$.

Solution. These quantities have no common monomial factor; but the monomial factor $3\,a^2$ common to all the terms of the first of them, and the factor x common to all the terms of the second, being suppressed in columns 1 and

2, give the first lines of the following form of the process, which is similar to that in art. 62.

Col. 1.	Col. 2.	Col. 3.
$2x^3+7ax^2-9a^3$	$4x^3+5a^2x+21a^3$	2
$14x^3+49ax^2-63a^3$	$4x^3+14ax^2-18a^3$	$-x$
$14x^3-5ax^2-39a^2x$	$-14ax^2+5a^2x+39a^3$	
$54ax^2+39a^2x-63a^3$	$-14x^2+5ax+39a^2$	-9
$18x^2+13ax-21a^2$	$-14x^2-21ax$	$-7x$
$126x^2+91ax-147a^2$	$26ax+39a^2$	
$126x^2-45ax-351a^2$	$2x+3a$	1
$136ax+204a^2$	$2x+3a$	
Ans. $2x+3a$.	. . .	

Column 3, in this form, is the line of quotients. The 1st line of col. 1 is first divided by that of col. 2, and the remainder is the 3d line of col. 2; this remainder, simplified by the suppression of the factor a, is the 4th line of col. 2, and is used to divide the 1st line of col. 1. The 2d line of col. 1 is the 1st line multiplied by 7 in order to render its first term divisible by the first term of the new divisor; the remainder of the division is the 4th line of col. 1, which is simplified in the 5th line by the suppression of the factor $3a$. The 6th line of col. 1 is the 5th line, multiplied by 7 in order to render its first term divisible by the first term of the divisor already used; for it is to be observed, that *a divisor should continue to be used until a remainder is obtained in which the first term ceases to be divisible by the first term of the divisor, that is, until the exponent of its leading letter is smaller than that in the first term of the divisor.* The remainder arising from the division of the 5th line of col. 1 by the 4th line of col. 2 is the 8th line of col. 1, which, reduced by the suppression of the factor $68a$ is the last line of col. 1. The remainder of the division of the 4th line of col. 2 and the last line of col. 1 is the 6th line of col. 2, which reduced by the suppression of the factor $13x$, is the

7th line of col. 2, being the same with the last line of col. 1. The remainder of the last division is therefore zero, and the last divisor $2x + 3a$ is the greatest common divisor.

2. Find the greatest common divisor of $21a^3bx^7 - 21a^4bx^6 - 168a^7bx^3$ and $14a^2b^3cx^4 - 14a^3b^3cx^3 + 28a^4b^3cx^2 - 42a^5b^3cx - 140a^6b^3c$.

Solution. Since $7a^2b$ is a monomial factor of the two given quantities, suppress it, and they become

$3ax^7 - 3a^2x^6 - 24a^5x^3$.

$2b^2cx^4 - 2ab^2cx^3 + 4a^2b^2cx^2 - 6a^3b^2cx - 20a^4b^2c$.

The greatest common divisor of these two quantities, found as in the preceding example, is $x - 2a$, which, multiplied by the common monomial factor $7a^2b$, gives $7a^2b(x - 2a)$ for the required greatest common divisor.

3. Find the greatest common divisor of $x^3 - a^3$ and $x^2 - a^2$. *Ans.* $x - a$.

4. Find the greatest common divisor of $5a^3 - 10a^2b + 15b^3$ and $3a^3 + 6a^2b + 6ab^2 + 3b^3$. *Ans.* $a + b$.

5. Find the greatest common divisor of $x^4 + x^3 + x^2 + x - 4$ and $x^4 + 2x^3 + 3x^2 + 4x - 10$. *Ans.* $x - 1$.

6. Find the greatest common divisor of $7ax^5 + 21ax^4 + 14ax$ and $3x^6 + 3x^5 + 3x^4 - 3x^2$. *Ans.* $x^2 + x$.

7. Find the greatest common divisor of $81a^4x^4 - 24a^7x$ and $3ax^7 - 2a^2x^6 + 3a^3x^3 - 2a^4x^4$. *Ans.* $3ax^2 - 2a^2x$.

8. Find the greatest common divisor of $x^3 + x - 10$ and $x^4 - 16$. *Ans.* $x - 2$.

65. *When there are several terms in the given polynomials, which contain the same power of the letter according to which the terms are arranged, these terms are to be united in one, as in art. 53, and the compound terms thus formed are to be treated as monomials.*

66. EXAMPLES.

1. Find the greatest common divisor of

$$\begin{array}{r|r|r|r|l} x^3 & y^5 - 2x^4 & y^4 - 3x^4 & y^3 + 2x^4 & y^2 \\ -9x & +2x^3 & +27x^2 & -18x^2 & \\ & +18x^2 & & & \\ & -18x & & & \end{array}$$

and

$$\begin{array}{r|r|r|r|l} x^2 & y^4 - 3x^3 & y^3 - 4x^3 & y^2 + 4x^3 & y \\ +3x & -7x^2 & -12x^2 & +12x^2 & \\ & +6x & & & \end{array}$$

Solution. The factor $(x^2 + 3x)y$ is a common factor of all the terms, and is therefore to be suppressed, in order to be multiplied by the greatest common divisor of the remaining polynomials. The polynomials thus become

$$\begin{array}{r|r|r|r|l} x & y^4 - 2x^2 & y^3 - 3x^2 & y^2 + 2x^2 & y \\ -3 & +8x & +9x & -6x & \\ & -6 & & & \end{array}$$

and

$$\begin{array}{r|l} y^3 - 3x & y^2 - 4xy + 4x. \\ +2 & \end{array}$$

The suppression of the factor $(x - 3)y$ in the first of these polynomials reduces it to

$$\begin{array}{r|l} y^3 - 2x & y^2 - 3xy + 2x, \\ +2 & \end{array}$$

by which the second is to be divided, and the rest of the process is as follows:

Col. 1.

$$\begin{array}{r|l} y^3 - 3x & y^2 - 4xy + 4x \\ +2 & \\ y^3 - 2x & y^2 - 3xy + 2x \\ +2 & \\ \hline \multicolumn{2}{r}{-xy^2 - xy + 2x} \\ \multicolumn{2}{r}{-y^2 - y + 2} \\ \multicolumn{2}{r}{-y^2 - 2y} \\ \hline \multicolumn{2}{r}{y + 2} \\ \multicolumn{2}{r}{y + 2} \end{array}$$

Col. 2.

$$\begin{array}{r|r|r|l} y^3 - 2x & \multicolumn{2}{l|}{y^2 - 3xy + 2x} & 1 \\ +2 & & & \\ y^3 + & \multicolumn{2}{l|}{y^2 - 2y} & -y \\ \hline -2x & y^2 - 3x & y + 2x & +2x \\ +1 & +2 & & -1 \\ -2x & y^2 - 2x & y + 4x & \\ +1 & +1 & -2 & \\ \hline & -x & y - 2x & \\ & +1 & +2 & -y \\ & & y + 2 & 1 \end{array}$$

Ans. $(y + 2)(x^2 + 3x)y = (x^2 + 3x)(y^2 + 2y)$.

The third line of col. 1 is the remainder of the division of the 1st line of col. 1 by the 1st line of col. 2; and this remainder, reduced by the suppression of the factor x is the 4th line of col. 1. The 5th line of col. 2 is the remainder of the division of the 1st line of col. 2 by the 4th line of col. 1, and this remainder, reduced by the suppression of the factors $-x+1$ is the last line of col. 2. The 4th line of col. 1 is exactly divisible by the last line of col. 2, and therefore the greatest common divisor is the product of (x^2+3x) y by $y+2$. *Ans.* $(x^2+3x)(y^2+2y)$.

2. Find the greatest common divisor of the polynomials $a^2+b^2+c^2+2ab+2ac+2bc$ and $a^2-b^2-c^2-2bc$. *Ans.* $a+b+c$.

3. Find the greatest common divisor of the polynomials

$$\begin{array}{r|l} a^4-2b^2 & a^2+b^4 \\ -2c^2 & -2b^2c^2 \\ & +c^4 \end{array} \quad \text{and} \quad \begin{array}{r|l} a^3+3ba^2+3b^2 & a+b^3 \\ -c^2 & -bc^2. \end{array}$$

Ans. $a^2+2ab+b^2-c^2$.

4. Find the greatest common divisor of the polynomials

$$\begin{array}{r|r|r|r|r|l} x & y^5-3x & y^4-x^2 & y^3+x^3 & y^2-x^3 & y \\ -1 & +3 & +3x & -x & +x^2 & \\ & & -2 & & & \end{array}$$

and

$$\begin{array}{r|r|r|r|l} x^2 & y^4-3x^2 & y^3+x^3 & y^2-x^3 & y \\ -1 & +3 & +2x^2 & +x & \\ & & -x & & \\ & & -2 & & \end{array}$$

Ans. $y(y-1)(x-1)$.

67. ***Problem.*** *To reduce fractions to a common denominator.*

Solution. Multiply both terms of each fraction by the product of all the other denominators.

For the value of each fraction is, from art. 55, not changed by this process; and as each of the denominators thus obtained is the product of all the denominators, the fractions are all reduced to the same denominator.

68. But fractions can be reduced to a common denominator which is smaller than their continued product, whenever their denominators have a common multiple less than this product. For, by art. 55,

Fractions may be reduced to a common denominator, which is a common multiple of their denominators, by multiplying both their terms by the quotients, respectively obtained from the division of the common denominator by their denominators.

69. *Corollary.* An entire quantity may, by the preceding article, be reduced to an equivalent fractional expression having any required denominator, by regarding it as a fraction, the denominator of which is unity.

70. EXAMPLES.

1. Reduce $\frac{3}{8}, \frac{2}{3}, \frac{5}{6}$, to the common denominator 24.

Ans. $\frac{9}{24}, \frac{16}{24}, \frac{20}{24}$.

2. Reduce $\frac{a^2 b}{c^2 d}, \frac{3 e}{2 c^2}, 8 a, \frac{3 a}{4 c d}$ to the common denominator $4 c^2 d$.

Ans. $\frac{4 a^2 b}{4 c^2 d}, \frac{6 e d}{4 c^2 d}, \frac{32 a c^2 d}{4 c^2 d}, \frac{3 a c}{4 c^2 d}$.

Common Denominator.

3. Reduce $\frac{a+b}{a-b}$, 1, $\frac{1}{a+b}$, $\frac{c+d}{a^2-b^2}$ to the common denominator a^2-b^2.

Ans. $\frac{a^2+2ab+b^2}{a^2-b^2}$, $\frac{a^2-b^2}{a^2-b^2}$, $\frac{a-b}{a^2-b^2}$, $\frac{c+d}{a^2-b^2}$.

71. ***Problem.*** ***To find the least common multiple of given quantities.***

Solution. ***When the given quantities are decomposed into their simplest factors, as is the case with monomials, their least common multiple is readily obtained; for it is obviously equal to the product of all the unlike factors, each factor being raised to a power equal to the highest power which it has in either of the given quantities.***

But the common factors can always be obtained from the process of finding the greatest common divisor.

72. EXAMPLES.

1. Find the least common multiple of $2a^3b^2cx$, $3a^5bc^3x^2$, $6acx = 2 \cdot 3acx$, $9c^7x^{10} = 3^2c^7x^{10}$, $24a^8 = 2^3 \cdot 3a^8$. *Ans.* $2^3 \cdot 3^2 \cdot a^8b^2c^7x^{10} = 72a^8b^2c^7x^{10}$.

2. Find the least common multiple of $16ax$, $40b^5x$, $25a^7b^3x^2$. *Ans.* $400a^7b^5x^2$.

3. Find the least common multiple of x^n, x^{n-1}, x^{n-2}, x^{n-3}, x. *Ans.* x^n.

4. Find the least common multiple of $6(a+b)x^m$, $54(a-b)^3$, $(a+b)^7$, $81(a-b)^3x^{m+2}$, $8(a+b)^5x^{m-8}$. *Ans.* $648(a+b)^7(a-b)^3x^{m+2}$.

5. Find the least common multiple of $a^2+2ab+b^2$, $a^2+4ab+4b^2$, a^2-b^2, $a^2+3ab+2b^2$, $a^3+a^2b-ab^2-b^3$. *Ans.* $(a+b)^2(a-b)(a+2b)^2$.

SECTION II.

Addition and Subtraction of Fractions.

73. *Problem. To find the sum or difference of given fractions.*

Solution. When the given fractions have the same denominator, their sum or difference is a fraction which has for its denominator the given common denominator, and for its numerator the sum or the difference of the given numerators.

When the given fractions have different denominators, they are to be reduced to a common denominator by arts. 67 *and* 68.

74. EXAMPLES.

1. Find the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $-\frac{e}{f}$.
Ans. $\frac{adf+bcf-bde}{bdf}$.

2. Subtract $\frac{a}{b}$ from $\frac{c}{d}$. *Ans.* $\frac{bc-ad}{bd}$.

3. Find the sum of $\frac{a+b}{2}$ and $\frac{a-b}{2}$ *Ans.* a.

4. Subtract $\frac{a-b}{2}$ from $\frac{a+b}{2}$. *Ans.* b.

Sum and Difference of Fractions.

5. Reduce to one fraction the expression $\frac{a}{b} + c$.

Ans. $\frac{a + bc}{b}$.

6. Reduce to one fraction $\frac{2a}{3bc} + \frac{5df}{8b^2c} - \frac{beg}{6b^2c^2}$.

Ans. $\frac{16abc + 15cdf - 4beg}{24b^2c^2}$.

7. Reduce to one fraction $\frac{a}{x} - \frac{b}{x^2} + \frac{1}{x^3}$.

Ans. $\frac{ax^2 - bx + 1}{x^3}$.

8. Reduce to one fraction $\frac{a}{a+z} + \frac{z}{a-z}$.

Ans. $\frac{a^2 + z^2}{a^2 - z^2}$.

9. Reduce to one fraction

$$\frac{3}{4(1-x)^2} + \frac{3}{8(1-x)} + \frac{1}{8(1+x)} - \frac{1-x}{4(1+x^2)}.$$

Ans. $\frac{1 + x + x^2}{1 - x - x^4 + x^5}$.

10. Reduce to one fraction

$$\frac{3h}{(h-2x)^2} + \frac{2h+x}{(h+x)(h-2x)} - \frac{5}{h+x}.$$

Ans. $\frac{20hx - 22x^2}{(h+x)(h-2x)^2}$.

11. Reduce to one fraction

$$\frac{a^3}{(a+b)^3} - \frac{ab}{(a+b)^2} + \frac{b}{a+b}.$$

Ans. $\frac{a^3 + ab^2 + b^3}{(a+b)^3}$.

75. *Corollary.* It follows, from examples 3 and 4, that the sum of half the sum and half the difference

of two quantities is equal to the greater of the two quantities; and that the difference of half their sum and half their difference is equal to the smaller of them.

SECTION III.

Multiplication and Division of Fractions.

76. *Problem. To find the continued product of several fractions.*

Solution. The continued product of given fractions is a fraction the numerator of which is the continued product of the given numerators, and the denominator of which is the continued product of the given denominators.

77. *Problem. To divide by a fraction.*

Solution. Multiply by the divisor inverted.

The preceding rules for the addition, subtraction, multiplication, and division of fractions require no other demonstrations than those usually given in arithmetic.

78. When the quantities multiplied or divided contain fractional terms, it is generally advisable to reduce them to a single fraction by means of art. 73

79. EXAMPLES.

1. Multiply together $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$. *Ans.* $\frac{a\,c\,e}{b\,d\,f}$.

2. Multiply $\frac{6\,a^3\,b^5}{7\,d^8\,e^9}$ by $\frac{3\,a^2\,d^3}{2\,b^5\,e}$. *Ans.* $\frac{9\,a^5}{7d^5\,e^{10}}$.

Product and Quotient of Fractions. Reciprocal.

3. Multiply $\frac{a}{b}$ by $\frac{b}{a}$. *Ans.* 1.

4. Multiply $a + \frac{ax}{a-x}$ by $x - \frac{ax}{a+x}$.

Ans. $\frac{a^2x^2}{a^2-x^2}$.

5. Divide $\frac{a}{b}$ by $\frac{c}{d}$. *Ans.* $\frac{ad}{bc}$.

6. Divide 1 by $\frac{a}{b}$. *Ans.* $\frac{b}{a}$.

7. Divide $\frac{15a^3x^5}{2d^4y^7}$ by $\frac{3a^6y^2}{8dx^7}$. *Ans.* $\frac{20x^{12}}{a^3d^3y^9}$.

8. Divide $x^2 + \frac{x^4}{a^2-x^2}$ by $\frac{ax}{a-x} - x$.

Ans. $\frac{a^2}{a+x}$.

80. The *reciprocal* of a quantity is the quotient obtained from the division of unity by the quantity.

Thus, the reciprocal of a is $\frac{1}{a}$ or a^{-1}, that of a^n is $\frac{1}{a^n}$ or a^{-n}, that of a^{-n} is a^n, and that of $\frac{a}{b}$ is $1 \div \frac{a}{b}$ or $\frac{b}{a}$.

Hence the product of a quantity by its reciprocal is unity; the reciprocal of a fraction is the fraction inverted; and the reciprocal of the power of a quantity is the same power with its sign reversed.

81. *Corollary.* To divide by a quantity is the same as to multiply by its reciprocal; and, conversely, to multiply by a quantity is the same as to divide by its reciprocal.

Now a fraction is multiplied either by multiplying its numerator or by dividing its denominator; and it is divided either by dividing its numerator or by multiplying its denominator. Hence,

It has the same effect to multiply one of the terms of a fraction by a quantity, which it has to multiply the other term by the reciprocal of the quantity.

82. *Corollary.* If either term of a fraction is multiplied by the power of a quantity, this factor may be suppressed, and introduced as a factor into the other term with the sign of the power reversed.

By this means, a fraction can be freed from negative exponents.

83. EXAMPLES.

1. Free the fraction $\frac{a^{-3}\,b^{2}}{d^{-1}\,e}$ from negative exponents.

   Ans. $\frac{b^{2}\,d}{a^{3}\,e}$.

2. Free the fraction $\frac{a^{5}\,b^{-3}\,c^{-2}}{a^{-2}\,c^{2}\,d}$ from negative exponents

   Ans. $\frac{a^{7}}{b^{3}\,c^{4}\,d}$.

3. Free the fraction $\frac{a^{-3}\,b^{2}\,c^{-5}\,d^{-1}\,e}{a\,b^{-3}\,c^{\ \ 2}\,d^{-8}\,f^{2}}$ from negative exponents.

   Ans. $\frac{b^{5}\,d^{7}\,e}{a^{4}\,c^{3}\,f^{2}}$.

4. Free the fraction $\frac{x^{3}+3\,x^{-2}}{x^{-1}+x^{-5}}$ from negative exponents.

   Ans. $\frac{x^{8}+3\,x^{3}}{x^{4}+1}$.

5. Free the fraction $\frac{1+x^{-2}+y^{-2}}{1+x^{-2}-y^{-2}}$ from negative exponents. *Ans.* $\frac{x^2y^2+y^2+x^2}{x^2y^2+y^2-x^2}$.

84. The preceding rules for fractions may all be applied to ratios by substituting the term *antecedent for numerator, and consequent for denominator.*

SECTION IV.

Proportions.

85. A ***proportion*** is the equation formed of two equal ratios.

Thus, if the two ratios $A:B$ and $C:D$ are equal, the equation

$$A:B=C:D$$

is a proportion; and it may also be written

$$\frac{A}{B}=\frac{C}{D}.$$

The first and last terms of a proportion are called its *extremes;* and the second and third its *means.*

Thus, A and D are the extremes of this proportion, and B and C its means.

86. If the ratios of the preceding proportion are reduced to a common consequent, in the same way in which fractions are, by art. 67, reduced to a common denominator, we have

$$A\times B:B\times D=B\times C:B\times D;$$

that is, $A \times D$ and $B \times C$ have the same ratio to $B \times D$, and are consequently equal, that is,

$$A \times D = B \times C,$$

or *the product of the means of a proportion is equal to the product of its extremes.*

This proposition is called the *test* of proportions, that is, *if four quantities are such that the product of the first and last of them is equal to the product of the second and third, these four quantities form a proportion.*

Demonstration. Let A, B, C, D be four quantities such that

$$A \times D = B \times C.$$

We have, by dividing $B \times D$,

$$A \times D : B \times D = B \times C : B \times D,$$

or, by reducing these ratios to lower terms, as in art. 40,

$$A : B = C : D;$$

that is, A, B, C, D form a proportion.

87. *Corollary.* If A, B, C, D form a proportion, we obtain from the preceding test

$$\begin{aligned} A : C &= B : D \\ B : A &= D : C \\ B : D &= A : C \\ D : C &= B : A, \text{ \&c.}; \end{aligned}$$

that is, *the terms of a proportion may be transposed in any way which is consistent with the application of the test.*

To find the Fourth Term of a Proportion.

88. *Problem. Given three terms of a proportion, to find the fourth.*

Solution. The following solution is immediately obtained from the test.

When the required term is an extreme, divide the product of the means by the given extreme, and the quotient is the required extreme.

When the required term is a mean, divide the product of the extremes by the given mean, and the quotient is the required mean.

89. EXAMPLES.

1. Given the three first terms of a proportion respectively A, B, C; find the fourth. *Ans.* $\frac{BC}{A}$.

2. Given the three first terms of a proportion respectively $2ab^2, 3a^2b, 6b^3$; find the fourth. *Ans.* $9ab^2$.

3. Given the three first terms of a proportion respectively a^m, a^n, a^p; find the fourth. *Ans.* a^{n+p-m}.

4. Given the first term of a proportion a^3b^2, the second $3a^8b^3$, the fourth $7ab$; find the third. *Ans.* $\frac{7}{3}a^{-4}$.

5. Given the first term of a proportion $6a^{m-2}b$, the third $15a^3b^5$, the fourth $40a^{-(m-1)}$; find the second.

Ans. $16a^{-4}b^{-4}$.

6. Given the three last terms of a proportion respectively $a^2-b^2, 2(a+b), a^2+2ab+b^2$; find the first.

Ans. $2(a-b)$.

90. When both the means of a proportion are the same quantity, this common mean is called the *mean proportional* between the extremes.

Thus, when

$$A : B = B : C,$$

B is a mean proportional between A and C.

91. If the test is applied to the preceding proportion it gives

$$B^2 = A \times C;$$

whence

$$B = \sqrt{A \times C};$$

that is, *the mean proportional between two quantities is the square root of their product.*

92. A succession of several equal ratios is called a *continued proportion.*

Thus,

$$A : B = C : D = E : F, \text{ \&c.}$$

is a continued proportion.

93. *Theorem. The sum of any number of antecedents in a continued proportion is to the sum of the corresponding consequents, as one antecedent is to its consequent.*

Demonstration. Denote the value of each of the ratios in the continued proportion of the preceding article by M, and we have

$$M = A : B = C : D = E : F, \text{ \&c.};$$

whence

$$A = B \times M$$
$$C = D \times M$$
$$E = F \times M, \text{ \&c.};$$

and the sum of these equations is

$$A + C + E + \text{\&c.} = (B + D + F + \text{\&c.}) \times M;$$

whence

$$\frac{A+C+E+\&c.}{B+D+F+\&c.}=M=\frac{A}{B}=\frac{C}{D}=\frac{E}{F}, \&c.$$

94. *Corollary. Either antecedent may be repeated any number of times in the above sum, provided its consequent is also repeated the same number of times.*

95. *Corollary. Either antecedent may be subtracted instead of being added, provided its consequent is also subtracted.*

96. *Corollary.* The application of these results to the proportion

$$A:B=C:D,$$

gives

$$A+C:B+D=A:B=C:D$$
$$A-C:B-D=A:B=C:D$$
$$mA+nC:mB+nD=A:B=C:D$$
$$mA-nC:mB-nD=A:B=C:D;$$

whence

$$A+C:B+D=A-C:B-D$$
$$mA+nC:mB+nD=mA-nC:mB-nD;$$

or, transposing the means as in art. 87,

$$A+C:A-C=B+D:B-D$$
$$mA+nC:mA-nC=mB+nD:mB-nD;$$

that is, *the sum of the antecedents of a proportion is to the sum of the consequents, as the difference of the antecedents is to the difference of the consequents, or as either antecedent is to its consequent.*

Likewise, *the sum of the antecedents is to their difference, as the sum of the consequents is to their difference.*

Moreover, *in finding these sums and differences, each antecedent may be multiplied by any number, provided its consequent is multiplied by the same number.*

97. *Corollary.* These rules may also be applied to the proportion

$$A : C = B : D$$

obtained from

$$A : B = C : D$$

by transposing its means, and give

$$A + B : C + D = A - B : C - D$$
$$= m A + n B : m C + n D = m A - n B : m C - n D$$
$$= A : C = B : D;$$

and

$$A + B : A - B = C + D : C - D$$
$$m A + n B : m A - n B = m C + n D : m C - n D;$$

that is, *the sum of the first two terms of a proportion is to the sum of the last two, as the difference of the first two terms is to the difference of the last two, or as the first term is to the third, or as the second is to the fourth.*

Likewise, *the sum of the first two terms is to their difference, as the sum of the last two is to their difference.*

Moreover, *in finding these sums and differences, both the antecedents may be multiplied by the same number, and both the consequents may be multiplied by any number.*

98. Two proportions, as

$$A : B = C : D$$

and

$$E : F = G : H,$$

may evidently *be multiplied together, term by term, and the result*

$$A \times E : B \times F = C \times G : D \times H$$

is a new proportion.

99. Likewise, *all the terms of a proportion may be raised to the same power.*

Thus, $A : B = C : D$

gives

$$A^2 : B^2 = C^2 : D^2$$
$$\sqrt{A} : \sqrt{B} = \sqrt{C} : \sqrt{D}$$
$$A^m : B^m = C^m : D^m$$
$$\sqrt[m]{A} : \sqrt[m]{B} = \sqrt[m]{C} : \sqrt[m]{D}$$
$$A^{-m} : B^{-m} = C^{-m} : D^{-m}.$$

100. *Theorem. The reciprocals of two quantities are in the inverse ratio of the quantities themselves.*

Thus $A : B = \frac{1}{B} : \frac{1}{A}$.

Demonstration. For A, B, $\frac{1}{B}$, and $\frac{1}{A}$ are four quantities such that the product of the first A and the last $\frac{1}{A}$ is the same with that of the second B and the third $\frac{1}{B}$; each product being equal to unity.

CHAPTER III.

EQUATIONS OF THE FIRST DEGREE.

SECTION I.

Putting Problems into Equations.

101. The first step in the algebraic solution of a problem is the expressing of its conditions in algebraic language; this is called *putting the problem into equations.*

102. No rule can be given for putting questions into equations, which is universally applicable. The following rule, can, however, be used in most cases, and problems, in which it will not succeed, must be considered as exercises for the ingenuity.

Represent the required quantities by letters of the alphabet. Perform or indicate upon these letters the same operations which it is necessary to perform upon their values, when obtained, in order to verify them.

It is usual to represent the unknown quantities by the last letters of the alphabet, as v, w, x, y, z.

103. EXAMPLES.

The following problems are to be put into equations.

1. A person had a certain sum of money before him From this he first took away the third part, and put in its

stead \$ 50; a short time after, from the sum thus increased he took away the fourth part, and put again in its stead \$ 70. He then counted his money, and found \$ 120. What was the original sum?

Method of putting into equations. Let

$$x = \text{the original sum expressed in dollars.}$$

After taking away the third part and putting in its stead \$ 50, there remains two thirds of the original sum increased by \$ 50, or

$$\tfrac{2}{3}\,x + 50.$$

If from this sum is taken a fourth part, there remains three fourths; to which is to be added \$ 70, giving

$$\tfrac{3}{4}\left(\tfrac{2}{3}\,x + 50\right) + 70 = \tfrac{1}{2}\,x + 107\tfrac{1}{2};$$

which is found to be equal to \$ 120. We have, therefore, for the required equation,

$$\tfrac{1}{2}\,x + 107\tfrac{1}{2} = 120.$$

2. A merchant adds yearly to his capital one third of it, but takes from it at the end of each year \$ 1000 for his expenses. At the end of the third year, after deducting the last \$ 1000, he finds himself in possession of twice the sum he had at first. How much did he possess originally?

Ans. If $x =$ the original capital in dollars, the required equation is

$$\tfrac{64}{27}\,x - 4111\tfrac{1}{9} = 2\,x.$$

3. A courier, who goes $31\tfrac{1}{2}$ miles every 5 hours, is sent from a certain place; when he was gone 8 hours, another was sent after him at the rate of $22\tfrac{1}{2}$ miles every 3 hours. How soon will the second overtake the first?

Solution. If $x =$ the required number of hours, the number of hours which the first courier is on the road is $x + 8$;

and the distance which he goes is obtained from the proportion

$$5 : x + 8 = 31\tfrac{1}{2} : \text{distance gone by 1st courier},$$

whence, by art. 88,

$$\text{distance gone by 1st courier} = \tfrac{63}{10}\,(x + 8).$$

The distance gone by the second courier is obtained from the proportion

$$3 : x = 22\tfrac{1}{2} : \text{distance gone by 2d courier};$$

whence

$$\text{distance gone by 2d courier} = \tfrac{45}{2}\,x.$$

But as both couriers go the same distance, the required equation is

$$\tfrac{63}{10}\,(x + 8) = \tfrac{45}{2}\,x.$$

4. A courier went from this place, n days ago, at the rate of a miles a day. Another has just started, in pursuit of him, at the rate of b miles a day. In how many days will the second courier overtake the first?

Ans. If $x =$ the required number of days, the required equation is

$$b\,x = a\,(x + n).$$

5. A regiment marches from the place A, on the road to B, at the rate of 7 leagues every 2 days; 8 days after, another regiment marches from B, on the road to A, at the rate of 31 leagues every 6 days. If the distance between A and B is 80 leagues, in how many days after the departure of the first regiment will the two regiments meet?

Ans. If $x =$ the required number of days, the required equation is

$$\tfrac{7}{2}\,x + \tfrac{31}{6}\,(x - 8) = 80.$$

Examples of putting Questions into Equations.

6. A hostile corps has set out two days ago from a certain place, and goes 27 miles daily. Another corps wishes to march in pursuit of it from the same place, and so quickly that it may reach the other in 6 days. How many miles must it march daily to accomplish it?

Ans. If $x =$ the required number of miles, the required equation is

$$6\,x = 216.$$

7. From two different sized orifices of a reservoir, the water runs with unequal velocities. We know that the orifices are in size as 5 : 13, and the velocities of the fluid are as 8 : 7; we know farther, that in a certain time there issued from the one 561 cubic feet more than there did from the other. How much water, then, did each orifice discharge in this space of time?

Solution. Let $x =$ the quantity discharged by the first orifice.

As the size of the second orifice is $\frac{13}{5}$ths of that of the first, the water discharged from the second orifice, if it flowed at the same rate, would be

$$\tfrac{13}{5}\,x.$$

But as the water flows from the second orifice with a velocity $\frac{7}{8}$ths of that which it should have to discharge $\frac{13}{5}\,x$ in the given time, its actual discharge must be

$$\tfrac{7}{8}\,(\tfrac{13}{5}\,x) = \tfrac{91}{40}\,x;$$

whence the required equation is

$$\tfrac{91}{40}\,x - x = 561.$$

8. A dog pursues a hare. When the dog started, the hare had made 50 paces before him. The hare takes 6 paces to the dog's five; and 9 of the hare's paces are equal to 7 of the dog's. How many paces can the hare take before the dog catches her?

5*

Ans. If $x =$ the required number of paces, the required equation is

$$\frac{14}{11} x - x = 50.$$

9. A work is to be printed, so that each page may contain a certain number of lines, and each line a certain number of letters. If we wished each page to contain 3 lines more, and each line 4 letters more, then there would be 224 letters more on each page; but if we wished to have 2 lines less in a page, and 3 letters less in each line, then each page would contain 145 letters less. How many lines are there in each page? and how many letters in each line?

Solution. Let

$x =$ the number of lines in a page;

$y =$ the number of letters in a line,

and we shall have

$x\,y =$ the number of letters in a page.

But if there were 3 lines more in a page, and 4 letters more in a line, the number of letters in a page would be

$$(x + 3)(y + 4) = x\,y + 4\,x + 3\,y + 12,$$

which exceeds the required number of letters in a page by

$$4\,x + 3\,y + 12\,;$$

whence we have for one of the required equations

$$4\,x + 3\,y + 12 = 224\,;$$

and, in the same way, the condition, that 2 lines less in a page and 3 letters less in a line make 145 letters less in a page, gives the equation

$$x\,y - (x - 2)(y - 3) = 145\,;$$

or

$$3\,x + 2\,y - 6 = 145.$$

10. Three soldiers, in a battle, make \$ 96 booty, which they wish to share equally. In order to do this, *A*, who made most, gives *B* and *C* as much as they already had; in

the same manner, B next divided with A and C, and after this, C with A and B. If, then, by these means, the intended equal division is effected, how much booty did each soldier make?

Ans. If $x = A$'s booty,
$y = B$'s booty,
$z = C$'s booty,

the required equations are

$$x + y + z = 96$$
$$4x - 4y - 4z = 6y - 2x - 2z$$
$$4x - 4y - 4z = 7z - x - y.$$

11. A certain number consists of three digits, of which the digit occupying the place of tens is half the sum of the other two. If this number be divided by the sum of its digits, the quotient is 48; but if 198 be subtracted from it, then we obtain for the remainder a number consisting of the same digits, but in an inverted order. What number is this?

Ans. If $x =$ the digit which is in the place of units;
$y =$ that in the place of tens,
$z =$ that in the place of hundreds.

The number is $= 100z + 10y + x$,

and the required equations are

$$y = \tfrac{1}{2}(x + z)$$
$$\frac{100z + 10y + x}{x + y + z} = 48$$
$$100z + 10y + x - 198 = 100x + 10y + z.$$

12. A person goes to a tavern with a certain sum of money in his pocket, where he spends 2 shillings; he then borrows as much money as he had left, and going to another tavern, he there spends 2 shillings also; then borrowing again as much money as was left, he went to a third tavern,

where likewise he spent 2 shillings, and borrowed as much as he had left; and again spending 2 shillings at a fourth tavern, he then had nothing remaining. What had he at first?

Ans. If $x =$ the shillings he had at first,

the required equation is

$$8x - 30 = 0.$$

13. A person possessed a certain capital, which he placed out at a certain interest. Another person, who possessed \$ 10 000 more than the first, and who put out his capital 1 per cent. more advantageously than the first did, had an income greater by \$ 800. A third person, who possessed \$ 15 000 more than the first, and who put out his capital 2 per cent. more advantageously than the first, had an income greater by \$ 1500. Required the capitals of the three persons, and the three rates of interest.

Ans. If $x =$ the capital of the first,
$y =$ his rate of interest per cent.

the required equations are

$$\frac{10\,000\,y + x + 10\,000}{100} = 800,$$

$$\frac{15\,000\,y + 2x + 30\,000}{100} = 1500.$$

14. A person has three kinds of goods, which together cost \$ $230\frac{5}{24}$. The pound of each article costs as many twenty-fourths of a dollar as there are pounds of that article; but he has one third more of the second kind than he has of the first, and $3\frac{1}{2}$ times as much of the third as he has of the second. How many pounds has he of each article?

Ans. If $x =$ the number of pounds of the first,

the required equation is

$$\tfrac{1}{24}x^2 + \tfrac{2}{27}x^2 + \tfrac{49}{54}x^2 = 230\tfrac{5}{24}.$$

Examples of putting Questions into Equations.

15. A person buys some pieces of cloth, at equal prices, for \$ 60. Had he got 3 pieces more for the same sum, each piece would have cost him \$ 1 less. How many pieces did he buy?

Ans. If $x =$ the number of pieces bought the required equation is

$$\frac{60}{x} = \frac{60}{x+3} + 1.$$

16. Two drapers A and B cut, each of them, a certain number of yards from a piece of cloth; A however 3 yards less than B, and jointly receive for them \$ 35. "At my own price," said A to B, "I should have received \$ 24 for your cloth." "I must admit," answered the other, "that, at my low price, I should have received for your cloth no more than \$ 12½." How many yards did each sell?

Solution. Let $x =$ the number of yards sold by A;

then $x + 3 =$ the number sold by B.

Now since A would have sold $x + 3$ yards for \$ 24,

$$A\text{'s price per yard} = \frac{24}{x+3};$$

and since B would have sold x yards for \$ 12½,

$$B\text{'s price per yard} = \frac{12\frac{1}{2}}{x} = \frac{25}{2x}.$$

Hence

$$\text{the sum for which } A \text{ sells } x \text{ yards} = \frac{24x}{x+3},$$

$$\text{the sum for which } B \text{ sells } x + 3 \text{ yards} = \frac{25(x+3)}{2x};$$

and the required equation is

$$\frac{24x}{x+3} + \frac{25(x+3)}{2x} = 35.$$

17. Two travellers, A and B, set out at the same time from two different places, C and D; A, from C to D; and B, from D to C. When they met, it appeared that A had already gone 30 miles more than B; and, according to the rate at which they are travelling, A calculates that he can reach the place D in 4 days, and that B can arrive at the place C in 9 days. What is the distance between C and D?

Ans. If, when they meet,

$$x = \text{the distance gone by } A,$$

then, $$x - 30 = \text{the distance gone by } B;$$

$$\text{the whole distance} = 2x - 30;$$

and the required equation is

$$\frac{4x}{x-30} = \frac{9(x-30)}{x}.$$

18. Some merchants jointly form a certain capital, in such a way that each contributes 10 times as many dollars as they are in number; they trade with this capital, and gain as many dollars per cent. as exceed their number by 8. Their profit amounts to \$ 288. How many were there of them?

Ans. If x = the number of merchants, the required equation is

$$\tfrac{1}{10}x^2(x+8) = 288.$$

19. Part of the property of a merchant is invested at such a rate of compound interest, that it doubles in a number of years equal to twice the rate per cent. What is the rate of interest?

Ans. If x = the rate per cent., the required equation is

$$\left(\frac{100+x}{100}\right)^{2x} = 2.$$

SECTION II.

Reduction and Classification of Equations.

104. The portions of an equation, which are separated by the sign $=$, are called its *members*; the one at the left of the sign being called its *first* member, and the other its *second* member.

105. Equations are divided into classes according to the form in which the unknown quantities are contained in them. But before deciding to which class an equation belongs, it should be freed from fractions, from negative exponents, and from the radical signs which affect its unknown quantities; its members should, if possible, be reduced to a series of monomials, and the polynomials thus obtained should be reduced to their simplest forms.

106. When the equation is thus reduced, it is said to be *of the same degree* as the number of dimensions of the unknown quantities in that term which contains the greater number of dimensions of the unknown quantities.

Thus, x and y being the unknown quantities, the equations

$$a\,x + b = c,$$
$$10\,x + y = 3,$$

are of the *first* degree;

$$x^2 + 3\,x + 1 = 5,$$
$$x\,y = 11,$$

are of the *second* degree, &c.

107. But when an equation does not admit of being reduced to a series of monomials, or, when being so reduced, it contains terms in which the unknown quantities or their powers enter otherwise than as factors, it is said to be *transcendental;* and the consideration of such equations belongs to the higher branches of mathematics.

Thus,
$$a^x = b$$
$$(x + a)^{y+b} = c,$$
are transcendental equations.

108. An equation is said to be *solved*, when the values of its unknown quantities are obtained; and these values are called the *roots* of the equation.

109. The reduction and solution of all equations depends upon the self-evident proposition, that

Both members of an equation may be increased, diminished, multiplied, or divided by the same quantity, without destroying the equality.

110. *Corollary. If all the terms of an equation have a common factor, this factor may be suppressed.*

111. EXAMPLES.

1. If the factor common to the terms of the equation
$$a^2 x^5 + 3 a^3 x^2 = a^2 x^2$$
is suppressed, what is the resulting equation?

Ans. $x^3 + 3a = 1.$

2. If the factor common to the terms of the equation
$$a^x + 3 a^{x+1} x = a^{x-1}$$
is suppressed, what is the resulting equation?

Ans. $a + 3 a^2 x = 1.$

112. ***Problem.*** ***To free an equation from fractions.***

Solution. ***Reduce, by arts.*** 67 ***and*** 68, ***all the terms of the equation to fractions having a common denominator, and suppress the common denominator, prefixing to the numerators the signs of their respective fractions.***

Demonstration. For suppressing the denominator of a fraction is the same as multiplying the fraction by its denominator; and, consequently, both the members of this equation are, by the preceding process, multiplied by the common denominator.

113. *Corollary.* It must be strictly observed that, when the denominator of a fraction is removed, the sign, which precedes the fraction, affects all the terms of the numerator. If therefore this sign is negative, all the signs of the numerator are to be reversed.

114. EXAMPLES.

1. Free the equation

$$\frac{a}{b\,x}+\frac{c}{d\,x}-\frac{a-c}{b\,d\,x}=h-\frac{1}{x}$$

from fractions.

Solution. This equation, when its terms are reduced to a common denominator, is

$$\frac{a\,d}{b\,d\,x}+\frac{b\,c}{b\,d\,x}-\frac{a-c}{b\,d\,x}=\frac{b\,d\,h\,x}{b\,d\,x}-\frac{b\,d}{b\,d\,x}.$$

Suppressing the common denominator, we have

$$ad+bc-(a-c)=bdhx-bd,$$

or

$$ad+bc-a+c=bdhx-bd.$$

2. Free the equation

$$\frac{3a-5x}{a-c}+\frac{2a-x}{d}=\frac{a+f}{a-c}-dx$$

from fractions.

Ans. $3ad-5dx+2a^2-ax-2ac+cx=ad+df-ad^2x+cd^2x.$

3. Free the equation

$$\frac{8x}{x+2}-6=\frac{20}{3x}$$

from fractions.

Ans. $24x^2-18x^2-36x=20x+40.$

4. Free the equation

$$\frac{18+x}{6(3-x)}=\frac{20x+9}{19-7x}-\frac{65}{4(3-x)}$$

from fractions.

Ans. $684-214x-14x^2=612x+324-240x^2-3705+1365x.$

5. Free the equation

$$\frac{x+y}{x-y}-\frac{x-y}{x+y}=\frac{1}{x-y}-\frac{1}{x+y}+\frac{1}{x^2-y^2}$$

from fractions.

Ans. $x^2+2xy+y^2-x^2+2xy-y^2=x+y-x+y+1.$

6. Free the equation

$$\frac{a^x}{b^x}=\frac{a^x+b^x}{a^x-b^x}$$

from fractions.

Ans. $a^{2x}-a^x b^x=a^x b^x+b^{2x}.$

To free a Fraction from negative Exponents.

115. *Corollary.* If the given equation contains negative exponents, it can be freed from them by arts. 80 and 82.

116. EXAMPLES.

1. Free the equation

$$\frac{x+x^{-1}}{x-x^{-1}}=x^{-1}$$

from fractions and negative exponents.

Ans. $x^3+x=x^2-1$.

2. Free the equation

$$\frac{x^a+x^{-a}}{a^x+a^{-x}}=\frac{a^x-a^{-x}}{x^a-x^{-a}}$$

from fractions and negative exponents.

Ans. $x^{4a}a^{2x}-a^{2x}=a^{4x}x^{2a}-x^{2a}$.

117. *Theorem. A term may be transposed from one member of an equation to the other member, by merely reversing its sign;* that is, it may be suppressed in one member and annexed to the other member with its sign reversed from + to —, or from — to +.

Proof. For suppressing it in the member in which it at first occurs is the same as subtracting it from that member; and annexing it to the other member with its sign reversed is, by art. 26, subtracting it from the other member; and, therefore, by art. 109, the equality is preserved.

118. *Corollary. All the terms of an equation may be transposed to either member, leaving zero in the other member; and the polynomial thus formed may be reduced to its simplest form, by arts.* 20 *and* 110.

119. EXAMPLES.

1. Reduce the equation

$$\frac{7\,x^n}{x-1}=\frac{6\,x^{n+1}+x^n}{x+1}-\frac{3\,x^n+6\,x^{n+2}}{x^2-1}$$

to its simplest form in a series of monomials.

Solution. This equation, freed from fractions by arts. 112 and 113, is

$$7\,x^{n+1}+7\,x^n=6\,x^{n+2}-5\,x^{n+1}-x^n-3\,x^n-6\,x^{n+2},$$

which becomes, by the transposition of its terms and by the reduction of art. 20,

$$12\,x^{n+1}+11\,x^n=0,$$

and, by striking out the factor x^n,

$$12\,x+11=0.$$

2. Reduce the equation

$$\frac{x^2+1}{x^2-1}-\frac{x-1}{(x+1)^2}=\frac{x+1}{x-1}$$

to its simplest form in a series of monomials.

Ans. $2\,x^2+1=0.$

3. Reduce the equation

$$\frac{a\,x^2+b\,x+c}{x^2+1}=\frac{a\,x^2-b\,x-c}{x^2-1}$$

to its simplest form.

Ans. $b\,x+c-a=0.$

4. Reduce the equation

$$\frac{a^n+a^{-n}}{x^n+x^{-n}}=\frac{a^n-a^{-n}}{x^n-x^{-n}}$$

to its simplest form.

Ans. $a^{2n}=x^{2n}.$

SECTION III.

Solution of Equations of the First Degree, with one unknown quantity.

120. *Theorem. Every equation of the first degree, with one unknown quantity, can be reduced to the form*

$$A\,x + B = 0;$$

in which A *and* B *denote any known quantities, whether positive or negative, and* x *is the unknown quantity.*

Proof. When an equation of the first degree with one unknown quantity is reduced, as in art. 118, its first member is composed of two classes of terms, one of which contains the unknown quantity, and the other does not. If the unknown quantity, which we may suppose to be x, is taken out as a factor from the terms in which it is contained, and its multiplier represented by A, the aggregate of the first class of terms is represented by $A\,x$; and the aggregate of the terms of the second class may be represented by B; whence the equation is represented by

$$A\,x + B = 0.$$

121. *Problem. To solve an equation of the first degree with one unknown quantity.*

Solution. Having reduced the given equation to the form

$$A\,x + B = 0,$$

transpose B to the second member by art. 117, and we have

$$A\,x = -B.$$

Dividing both members of this equation by A, gives

$$x = -\frac{B}{A}.$$

Hence, *to solve an equation of the first degree, reduce it, as in art.* 120, *and transpose its known terms to the second member, and all its unknown terms to the first member; and the value of the unknown quantity is equal to the quotient arising from the division of the second member by the multiplier of the unknown quantity in the first member.*

122. *Corollary.* When A and B are both positive or both negative, the value of x is, by art. 35, negative; but when A and B are unlike in their signs, one positive and the other negative, x is positive.

123. *Corollary.* When we have

$$B = 0,$$

the value of x is

$$x = -\frac{0}{A} = 0.$$

124. *Corollary.* When we have

$$A = 0,$$

the value of x is

$$x = -\frac{B}{0}.$$

But the smaller a divisor is, the oftener must it be contained in the dividend, that is, the larger must the quotient be; and when the divisor is zero, it must be contained an infinite number of times in the dividend, or the quotient must be infinite. *Infinity* is represented by the sign ∞ We have, then, in this case,

$$x = -\infty$$

The given equation is, however, in this case,

$$0 \times x + B = 0,$$

which reduces itself to

$$B = 0,$$

an obvious absurdity, unless B is zero.

The sign ∞ is, therefore, rather to be regarded as the expression of the peculiar species of absurdity which arises from diminishing the denominator of a fraction till it becomes zero.

125. *Corollary.* When we have

$$A = 0, \text{ and } B = 0,$$

the value of x is

$$x = -\frac{0}{0} = \frac{0}{0},$$

which is equal to any quantity whatever, and is called an *indeterminate expression.*

The given equation is, indeed, in this case

$$0 \times x + 0 = 0,$$

an equation which is satisfied by any value whatever of x, and is called an *identical equation.*

126. EXAMPLES.

1. Solve the equation

$$8x - 5 = 13 - 7x.$$

Ans. $x = 1\frac{1}{5}$.

2. Solve the equation

$$\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 7x - 712 + \frac{x}{5}.$$

Ans. $x = 116\frac{148}{367}$.

3. Solve the equation

$$a x + c = b x + d.$$

Ans. $x = \dfrac{d - c}{a - b}.$

4. Solve the equation

$$\frac{a(d^2 + x^2)}{d x} = a c + \frac{a x}{d}.$$

Ans. $x = \dfrac{d}{c}.$

5. Solve the equation

$$\frac{c x^m}{a + b x} = \frac{f x^m}{d + e x}.$$

Ans. $x = \dfrac{c d - a f}{b f - c e} = \dfrac{a f - c d}{c e - b f}.$

6. Solve the equation

$$\frac{3 a b c}{a + b} + \frac{a^2 b^2}{(a + b)^3} + \frac{(2 a + b) b^2 x}{a (a + b)^2} = 3 c x + \frac{b x}{a}.$$

Ans. $x = \dfrac{a b}{a + b}.$

7. Two capitalists calculate their fortunes, and it appears that one is twice as rich as the other, and that together they possess \$ 38 700. What is the capital of each?

Ans. The one has \$ 12 900, the other \$ 25 800.

8. To find two such numbers, that the one may be m times as great as the other, and that their sum $= a$.

Ans. $\dfrac{a}{m + 1}$ and $\dfrac{m a}{m + 1}.$

9. The sum of \$ 1200 is to be divided between two persons, *A* and *B*, so that *A*'s share is to *B*'s as 2 to 7. How much does each receive?

Ans. *A* \$ $266\frac{2}{3}$, *B* \$ $933\frac{1}{3}$.

Equations of the First Degree with one unknown quantity.

10. To divide a number a into two such parts, that the first part is to the second as m to n.

Ans. $\frac{ma}{m+n}$ and $\frac{na}{m+n}$.

11. How much money have I, when the 4th and 5th parts of it amount together to \$2,25?

Ans. \$5.

12. Find a number such, that when it is divided successively by m and by n, the sum of the quotients $= a$.

Ans. $\frac{mna}{m+n}$.

13. Divide the number 46 into two parts, so that when the one is divided by 7, and the other by 3, the sum of the quotients $= 10$. *Ans.* 28 and 18.

14. All my journeyings taken together, says a traveller, amount to 3040 miles; of which I have travelled $3\frac{1}{2}$ times as much by water as on horseback, and $2\frac{1}{3}$ times as much on foot as by water. How many miles did he travel in each of these three ways?

Ans. 240 miles on horseback, 840 miles by water, and 1960 miles on foot.

15. Divide the number a into three such parts, that the second may be m times, and the third n times as great as the first.

Ans. $\frac{a}{1+m+n}$, $\frac{ma}{1+m+n}$, $\frac{na}{1+m+n}$.

16. A bankrupt leaves \$21 000 to be divided among four creditors A, B, C, D, in proportion to their claims. Now A's claim is to B's as $2:3$; B's claim : C's $= 4:5$; and C's claim : D's $= 6:7$. How much does each creditor receive?

Ans. A \$3200, B \$4800, C \$6000, D \$7000.

17. Divide the number a into three such parts, that the 1st shall be to the 2d as m to n; and the 2d part : the 3d $= p : q$.

$$Ans. \ \frac{mpa}{mp+np+nq}, \ \frac{npa}{mp+np+nq}, \ \frac{nqa}{mp+np+nq}.$$

18. There are two numbers whose sum is 96, and difference 16; what are they? *Ans.* 56 and 40.

19. A father gives to his five sons \$1000, which they are to divide according to their ages, so that each elder son shall receive \$20 more than his next younger brother. What is the share of the youngest? *Ans.* 160.

20. One has six sons, each whereof is 4 years older than his next younger brother; and the eldest is three times as old as the youngest. What is the age of the eldest?

Ans. 30 years.

21. There is a certain fish whose head is 9 inches; the tail is as long as the head and half the back; and the back is as long as both the head and the tail together. What is the length of the fish?

Ans. 72 inches.

22. Five gamesters have lost jointly \$$40\frac{5}{8}$; B's loss amounts to $\frac{1}{2}$ dollar more than triple A's; C's loss is \$2 less than twice B's; D lost $\frac{1}{4}$ dollar less than A and B together; and E twice as much as B less $\frac{1}{8}$ dollar. How much did each of them lose?

Ans. A \$2, B \$$6\frac{1}{2}$, C \$11, D \$$8\frac{1}{4}$, E \$$12\frac{7}{8}$.

23. A mason, 12 journeymen, and 4 assistants, receive together \$72 wages for a certain time. The mason receives \$1 daily, each journeyman $\frac{1}{2}$ dollar, and each assistant $\frac{1}{4}$ dollar. How many days must they have worked for this money? *Ans.* 9 days.

24. Find a number such that if you multiply it by 5, subtract 24 from the product, divide the remainder by 6, and add 13 to the quotient, you will obtain this number.

Ans. 54.

25. A courier left this place n days ago, and makes a miles daily. He is pursued by another making b miles daily. In how many days will the second overtake the first?

Ans. $\frac{n\,a}{b-a}$ days.

26. A courier started from a certain place 12 days ago, and is pursued by another, whose speed is to that of the first as 8 : 3. In how many days will the second overtake the first?

Ans. $7\frac{1}{5}$ days.

27. A courier started from this place n days ago, and is pursued by another whose speed is to that of the first as p is to q. In how many days will the second overtake the first?

Ans. $\frac{n\,q}{p-q}$.

28. Two bodies move in opposite directions; one moves c feet in a second, the other C feet. The two places, from which they start at the same time, are distant a feet from one another. When will they meet?

Ans. In $\frac{a}{C+c}$ seconds.

29. Two bodies move in the same direction from two places at a distance of a feet apart; the one at the rate of c feet in a second, the other pursuing it at the rate of C feet in a second When will they meet?

Ans. In $\frac{a}{C-c}$ seconds.

30. At 12 o'clock, both hands of a clock are together. When and how often will these hands be together in the next 12 hours?

Ans. At $5\frac{5}{11}$ minutes past 1, at $10\frac{10}{11}$ minutes past 2, at $16\frac{4}{11}$ minutes past 3, and so on, in each successive hour, $5\frac{5}{11}$ minutes later.

31. Two bodies move after one another in the circumference of a circle, which measures p feet. At first they are distant from each other by an arc measuring a feet, the first moves c feet, the second C feet, in a second. When will those two bodies meet for the first time, second time, and so on, supposing that they do not disturb each other's motion?

Ans. In $\frac{a}{C-c}, \frac{p+a}{C-c}, \frac{2p+a}{C-c}$, &c., seconds.

32. When will they meet if the first begins to move t seconds sooner than the second?

Ans. In $\frac{a+ct}{C-c}, \frac{p+a+ct}{C-c}, \frac{2p+a+ct}{C-c}$, &c., seconds.

33. But when will they meet, if the first begins to move t seconds later than the second?

Ans. In $\frac{a-ct}{C-c}, \frac{p+a-ct}{C-c}, \frac{2p+a-ct}{C-c}$, &c., seconds.

34. When will they meet, if the first, instead of running in the same direction with the second, runs in the opposite direction, and starts at the same time?

Ans. In $\frac{a}{C+c}, \frac{p+a}{C+c}, \frac{2p+a}{C+c}, \frac{3p+a}{C+c}$, &c., seconds.

35. When will they meet, if, moving in an opposite direction to the second, the first starts t seconds sooner than the second?

Ans. In $\frac{a-ct}{C+c}, \frac{p+a-ct}{C+c}, \frac{2p+a-ct}{C+c}$, &c., seconds.

Equations of the First Degree with one unknown quantity.

36. But when will they meet, if, moving in an opposite direction to the second, the first starts t seconds later than the second?

Ans. In $\frac{a+ct}{C+c}$, $\frac{p+a+ct}{C+c}$, $\frac{2p+a+ct}{C+c}$, &c., seconds.

37. A wine merchant has two kinds of wine; the one costs 9 shillings per gallon, the other 5. He wishes to mix both wines together, in such quantities, that he may have 50 gallons, and each gallon, without profit or loss, may be sold for 8 shillings. How must he mix them?

Ans. $37\frac{1}{2}$ gallons of the wine at 9 shillings, with $12\frac{1}{2}$ gallons of that at 5 shillings.

38. A wine merchant has two kinds of wine; the one costs a shillings per gallon, the other b shillings. How must he mix both these wines together, in order to have n gallons, at a price of c shillings per gallon?

Ans. $\frac{(a-c)n}{a-b}$ gallons of the wine at b shillings, and $\frac{(c-b)n}{a-b}$ gallons of that at a shillings.

39. To divide the number a into two such parts, that, if the first is multiplied by m and the second by n, the sum of the products is b.

Ans. $\frac{b-na}{m-n}$ and $\frac{ma-b}{m-n}$.

40. One of my acquaintances is now 30, his younger brother 20; and consequently $3:2$ is the ratio of his age to his brother's. In how many years will their ages be as $5:4$? *Ans.* In 20 years.

41. What two numbers are those, whose ratio $= a:b$; but, if c is added to both of them the resulting ratio $= m:n$?

Ans. $\frac{ac(m-n)}{an-bm}$ and $\frac{bc(m-n)}{an-bm}$.

42. Find a number such that 5 times the number is as much above 20, as the number itself is below 20.

Ans. $6\frac{2}{3}$.

43. A person wished to buy a house, and in order to raise the requisite capital, he draws the same sum from each of his debtors. He tried, whether, if he obtained \$250 from each, it would be sufficient for the purpose; he found, however, that he should then still lack \$2000. He tried it, therefore, with \$340; but this gave him \$880 more than he required. How many debtors had he?

Ans. 32.

44. A father leaves a number of children, and a certain sum, which they are to divide amongst them as follows: The first is to receive \$100, and then the 10th part of the remainder; after this, the second has \$200, and the 10th part of the remainder; again, the third receives \$300, and the 10th part of the remainder; and so on, each succeeding child is to receive \$100 more than the one preceding, and then the 10th part of that which still remains. But it is found that all the children have received the same sum. What was the fortune left? and what was the number of children?

Ans. The fortune was \$8100, and the number of children 9.

45. Divide the number 10 into two such parts, that the difference of their squares may be 20. *Ans.* 6 and 4.

46. Divide the number a into two such parts, that the difference of their squares may be b.

Ans. $\frac{a^2+b}{2a}$ and $\frac{a^2-b}{2a}$.

47. What two numbers are they whose difference is 5 and the difference of whose squares is 45?

Ans. 7 and 2.

Examples of unknown quantity equal to Zero.

48. What two numbers are they whose difference is a, and the difference of whose squares is b?

Ans. $\frac{b-a^2}{2a}$ and $\frac{b+a^2}{2a}$.

127. *Corollary.* When the solution of a problem gives zero for the value of either of the unknown quantities, this value is sometimes a true solution; and sometimes it indicates an impossibility in the proposed question. In any such case, therefore, it is necessary to return to the data of the problem and investigate the signification of this result.

128. EXAMPLES.

1. In what cases would the value of the unknown quantity in example 25 of art. 126 become zero? and what would this value signify?

Solution. As the value of the unknown quantity of the example is the fraction, which is its answer; it is zero, when

$$\frac{n\,a}{b-a}=0$$

or, clearing from fractions, when

$$n\,a=0;$$

that is, when

$$n=0, \text{ or when } a=0;$$

and, in either case, this value signifies that the couriers are together at the outset; and zero must, therefore, be regarded as a real solution.

2. In what cases would the value of the unknown quantity in example 35 of art. 126 become zero? and what would this value signify?

Ans. When $t = \frac{a}{c}$, or $= \frac{p+a}{c}$, or $= \frac{2p+a}{c}$, &c., and either of these equations signifies that the bodies are together when the second body starts, the first body having just arrived at the point of departure of the second, and zero is, therefore, to be regarded as a real solution

3. In what cases would the value of one of the unknown quantities in example 38 of art. 126 become zero? and what would this value signify?

Ans. When either

$$a = c, \text{ or } b = c;$$

and, in either case, these equations indicate that the price of one of the wines is just that of the required mixture, and, of course, needs *none* of the other wine added to it to make it of the required value; and zero, must, therefore, be regarded as a true solution.

4. In what cases would the value of one of the unknown quantities in example 39 of art. 126 become zero? and what would this value signify?

Ans. When

$$b = n\,a, \text{ or } = m\,a;$$

and these equations indicate that a is itself such that, multiplied either by m or by n, it gives a product $= b$; and zero may be regarded as a true solution, expressing that one of the parts is zero, while the other is the number a itself.

5. In what cases would the value of one of the unknown quantities in example 41 of art. 126 become zero? and what would this value signify?

Ans. First. When

$$a = 0, \text{ or } b = 0,$$

and, in this case, zero is a true solution by regarding all numbers as having the same ratio to zero.

Cases in which the value of an unknown quantity is infinite.

Secondly. When $c = 0$,
and, in this case, the problem is impossible, for no two numbers can be in the ratio $a : b$, and, without having any thing added to or subtracted from them, acquire the different ratio $m : n$.

Thirdly. When $m = n$,
and, in this case, the problem is impossible, for no two numbers, whose ratio $= a : b$, and which are therefore unequal, can, by the addition of c to each of them, become equal to each other, as required by the ratio $m : n = m : m = 1$.

129. When the solution of a problem gives, for the values of one of its unknown quantities, any fractions, the denominators of which are zero, while the numerators are not zero; such values are, generally, to be regarded as indicating an absurdity in the enunciation of the problem.

130. EXAMPLES.

1. In what case does the denominator of the fractional value of the unknown quantity in example 25 of art. 126 become zero? and what is the corresponding absurdity in the enunciation of the problem?

Ans. When $a = b$,
and the absurdity is, that, while the couriers are travelling at the same rate, it is required to determine the time in which one will overtake the other.

2. In what case do the denominators of the fractional values of the unknown quantity in example 38 of art. 126 become zero? and what is the corresponding absurdity in the enunciation of the problem?

Ans. When $a = b$,

and the absurdity is that, while both the wines are of the same value, they should give a mixture of a value different from their common value.

3. In what case would the denominators of the fractional values of the unknown quantities in example 41 of art. 126 become zero? and what is the corresponding absurdity of the enunciation?

Ans. When

$$a n = b m, \text{ that is, when } a : b = m : n;$$

and the absurdity is, that the ratio of two unequal numbers should not be changed by increasing them both by the same quantity.

4. In what case would the denominators of the fractional values of the unknown quantities in example 48 of art. 126 become zero? and what is the corresponding absurdity of the enunciation?

Ans. When $a = 0$,

and the absurdity is, that the squares of two equal numbers should differ.

131. *Corollary.* When the solution of a problem gives for the value of either of its unknown quantities a fraction whose terms are each equal to zero, this value generally indicates that the conditions of the problem are not sufficient to determine this unknown quantity, and that it may have any value whatever. In some cases, however, there are limitations to the change of value of the unknown quantity.

Cases in which the value of an unknown quantity is indeterminate.

132. EXAMPLES.

1. In what case would both the terms of the fractional value of the unknown quantity in example 25 of art. 126 become zero? and how could this value be a solution?

Ans. When $b = a$, and $n = 0$;

and these equations signify, that the couriers travel equally fast, and start at the same time; and, therefore, they remain together, and any number whatever may be taken as the value of the unknown quantity.

2. In what case would both the terms of either of the fractional values of the unknown quantity in example 31 of art. 126 become zero? and how could this value be a solution?

Ans. When $a = 0$, and $C = c$;

and these equations signify, that the bodies move equally fast, and start from the same place; they, therefore, remain together, and any number whatever may be taken as the value of the unknown quantity.

But, in this case, all the algebraic values of the unknown quantity but the first become infinite, as they should, because they are obtained on the supposition, that the second body has passed round the circle once, twice, &c., oftener than the first body; which is here impossible.

3. In what case would all the terms of the fractiona. values of the unknown quantities in example 38 of art. 126 become zero? and how could they, then, satisfy the conditions of the problem?

Ans. When $a = b = c$;

and these equations signify, that the wines and the mixture are all of the same value; in whatever proportion, therefore, the wines are mixed together, the mixture

must be of the required value. But the values of the unknown quantities are still subject to the limitation that their sum is n.

4. In what case would the terms of the fractional values of the unknown quantities in example 39 of art. 126 become zero? and how could they, then, satisfy the conditions of the problem?

Ans. When

$$m = n, \text{ and } b = n\,a = m\,a;$$

and these equations signify, that the sum b of the products of the parts of a multiplied by $m = n$ is to be equal to the product of a multiplied by n; and this is, evidently, the case into whatever parts a is divided.

5. In what cases would all the terms of the fractional values of the unknown quantities in example 41 of art. 126 become zero? and how could they, then, satisfy the conditions of the problem?

Ans. First. When

$$a : b = m : n, \text{ and } c = 0;$$

for these equations indicate that the two required numbers are only subject to the condition that their ratio $= a : b$.

Secondly. When

$$m = n, \text{ and } a : b = m : n = m : m = 1, \text{ that is, } a = b;$$

for these equations indicate that the two numbers are to be equal; and that they are to remain equal, when they are increased by c, which would always be the case.

6. In what case would all the terms of the fractional values of the unknown quantities in example 48 of art. 126 become zero? and how could these values be solutions?

Ans. When $a = 0$, and $b = 0$;

and their equations indicate that the numbers are to be equal, and that their squares are to be equal, which is always the case with equal numbers.

133. *Corollary.* When the solution of a problem gives a negative value to either of the unknown quantities, this value is not generally a true solution of the problem; and if the solution gives no other than negative values for this quantity, the problem is generally impossible.

But, in this case, the negative of the negative value of the unknown quantity is positive; so that the enunciation of the problem can often be corrected by changing it, so that this unknown quantity may be added instead of being subtracted, and the reverse.

134. EXAMPLES.

1. In what case would the value of the unknown quantity in example 25 of art. 126 be negative? why should it be so? and could the enunciation be corrected for this case?

Ans. When $a > b$;

that is, when the second courier goes slower than the one he is pursuing, in which case he evidently cannot overtake him; and the enunciation does not, in this case, admit of a legitimate correction.

2. In what case would the values of the unknown quantities in examples 29, 31, 32 of art. 126 be negative? why should this be so? and could the enunciations be corrected for this case?

Ans. When $c > C$;

that is, when the first body moves faster than the second, in which case the second cannot overtake it.

The enunciation may be corrected for this case by supposing the bodies to travel in the opposite direction to

Cases of negative value of unknown quantity.

that which they are at present taking, that is, by supposing the first body to pursue the second.

Examples 31 and 32 are not, however, impossible in this case; for, from the very nature of their circular motion, the first body is necessarily pursuing the second even in their present direction; the second body must not, however, be considered as a feet or $a + c\,t$ feet behind the first, but as $p - a$ or $p - (a + c\,t)$ feet before it.

3. In what cases would the values of the unknown quantity in example 33 of art. 126 be negative? why should this be the case? and could the enunciation be corrected for this case?

Ans. First. When $C < c$,
which is subject to the same remarks as in the preceding question.

Secondly. When $C > c$,

and $c\,t > a$, or $> p + a$, or $> 2p + a$, &c.,

that is, when the first body does not start until the second body has passed it once, or twice, or three times, &c.; and if the bodies were moving in the same straight line, the enunciation would not admit of legitimate correction. As it is, however, the first body is still pursued by the second, and is $p + a - c\,t$, $2p + a - c\,t$, &c., feet before the second, when it starts; so that all the values given for the unknown quantity are correct, except the negative ones.

4. In what cases would the values of the unknown quantity in example 35 of art. 126 be negative? why should this be the case? and could the enunciation be corrected for this case?

Ans. When

$c\,t > a$, or $> p + a$, or $> 2p + a$, &c.;

that is, when the first body has passed the second once, twice, &c., before the second begins to move.

If the bodies were moving in the same straight line,

the second body would be obliged to change its direction, and move in the same direction with the first, and even with this change of enunciation the problem is impossible, if the second body moves slower than the first.

But as it is, the bodies are still moving towards each other in the circumference of the circle; their distance apart at the instant when the second body starts being $p + a - c\,t$, or $2\,p + a - c\,t$, &c., feet; so that all the positive values of the unknown quantity remain as true solutions.

5. In what cases would the values of either of the unknown quantities in example 38 of art. 126 be negative? why should this be the case? and could the enunciation be corrected for this case?

Ans. If we suppose, as we evidently may, that $a > b$; one of the values is negative,

First. When $a < c$;

that is, when the price of the most expensive wine is less than that of the required mixture.

Secondly. When $b > c$;

that is, when the price of the least expensive wine is more than that of the mixture.

In either case the problem is altogether impossible, for two wines cannot be mixed together so as to produce a wine more valuable than either of them without a gain, or less valuable than either of them, without a loss.

6. In what cases would the value of either of the unknown quantities in example 39 of art. 126 be negative? why should this be so? and could the enunciation be corrected for this case?

Ans. Supposing, as we may, that $m > n$;

First. When $n\,a > b$,

that is, when the sum b of the products is less than the product of a by the least of the numbers m and n.

Cases of negative value of unknown quantity.

Secondly. When $m a < b$;

that is, when the sum b of the products is greater than the product of a by the greater of the numbers m and n.

In either of these cases, the problem is plainly impossible; and, in the corrected enunciation, a should be the difference of the required numbers, and b the difference of the products obtained from multiplying one of the numbers by m and the other by n.

7. In what cases would the values of the unknown quantities in example 41 of art. 126 be negative? why should this be so? and could the enunciation be corrected for this case?

Ans. First. When

$$m > n, \text{ and } a\,n < b\,m, \text{ or } a : b < m : n;$$

that is, when the first ratio is less than the second, and the second is greater than unity.

Secondly. When

$$m < n, \text{ and } a : b > m : n;$$

that is, when the second ratio is less than the first, and also less than unity.

In either case the problem is impossible, and c is to be subtracted instead of being added in the corrected enunciation.

8. In what case would the value of one of the unknown quantities in example 46 of art. 126 be negative? why should this be so? and could the enunciation be corrected for this case?

Ans. When $b > a^2$;

that is, when the difference of the squares of the parts of a is to be greater than the square of the number itself, which can never be the case; for the greatest possible difference of squares corresponds to the case in which one of the parts is the number a itself, and the other is zero; and

the difference of the squares is then just equal to the square of a.

The enunciation is corrected for this case by stating it as in example 48.

135. *Corollary.* It follows from example 7 of the preceding section that a fraction or ratio, which is greater than unity, is increased by diminishing both its terms by the same quantity; and a fraction or ratio, which is less than unity, is diminished by diminishing both its terms by the same quantity; but the reverse is the case, when the terms are increased instead of being diminished.

SECTION IV.

Equations of the First Degree containing two or more unknown quantities.

136. In the solution of complicated problems involving several equations, it is often found convenient to use the same letter to denote similar quantities, accents or numbers being placed to its right or left, above or below, so as to distinguish its different values.

$$
\begin{array}{llllll}
\text{Thus, } a, & a', & a'' & a''', & a^{\text{IV}}, \ldots & a^{(n)}, \ \&\text{c.} \\
a^{(1)}, & a^{(2)}, & a^{(3)}, & a^{(4)}, \ldots & a^{(n)}, \ \&\text{c.} \\
a_1, & a_2, & a_3, & a_4, \ldots & a_n, \ \&\text{c.} \\
{}'a, & {}''a, & {}'''a, & {}^{\text{IV}}a, \ldots\ldots & {}^{n}a, \ \&\text{c.} \\
{}^{1}a, & {}^{2}a, & {}^{3}a, & {}^{4}a, \ldots\ldots & {}^{n}a, \ \&\text{c.} \\
{}_{1}a, & {}_{2}a, & {}_{3}a, & {}_{4}a, \ldots\ldots & {}_{n}a, \ \&\text{c.} \\
a_2^1, & {}^{3}a_4, & {}_{1}a_3, & {}_{2}a^{\text{v}}, & {}_{1}^{3}a'', \ldots\ldots & {}_{n}^{n}a_n^{(n)}, \ \&\text{c.}
\end{array}
$$

may all be used to denote different quantities, though they generally are supposed to imply some similarity between the

quantities which they represent. Care must be taken not to confound the accents and the numbers in parentheses at the right with exponents.

137. *Problem. To solve an equation with several unknown quantities.*

Solution. Solve the given equation precisely as if all its unknown quantities were known, except any one of them which may be chosen at pleasure; and in the value of this unknown quantity, which is thus obtained in terms of the other unknown quantities, any values whatever may be substituted for the other unknown quantities, and the corresponding value of the chosen unknown quantity is thus obtained.

138. *Corollary.* An equation which contains several unknown quantities is not, therefore, sufficient to determine their values, and is called *indeterminate.*

139. *Scholium.* The roots of an indeterminate equation are sometimes subject to conditions which cannot be expressed by equations, and which limit their values; such, for instance, as that they are to be whole numbers. But their investigation depends, in such cases, upon the particular properties of different numbers, and belongs, therefore, to the *Theory of Numbers.*

140. *Theorem. Every equation of the first degree can be reduced to the form*

$$A x + B y + C z + \&c. + M = 0;$$

in which A, B, C, *&c. and* M *are known quantities,*

either positive or negative, and x, y, z, *&c. are the unknown quantities.*

Proof. . When an equation of the first degree is reduced, as in art. 118, the aggregate of all its known terms may be denoted by M. Each of the other terms must have one of the unknown quantities as a factor; and, by art. 106, only one of them, and that one taken but once as a factor. Taking out, then, each unknown quantity as a factor from the terms in which it occurs, and representing its multiplier by some letter, as A, B, C, &c., the corresponding unknown quantities being represented by x, y, z, &c., the equation becomes

$$A\,x + B\,y + C\,z + \&c. + M = 0.$$

141. *Problem. To solve any equation of the first degree.*

Solution. Having reduced the equation to the form

$$A\,x + B\,y + C\,z + \&c. + M = 0,$$

find, as in art. 137, the value of either of the unknown quantities, as x, for instance, which is, by art. 121,

$$x = \frac{-B\,y - C\,z - \&c. - M}{A},$$

and any quantities at pleasure may be substituted for y, z, &c.

142. *Problem. To solve several equations with several unknown quantities.*

First Method of Solution called that of Elimination by Substitution. Find the value of either of the unknown quantities in one of the equations in which it occurs, and substitute its value thus found, which is generally in terms of the other unknown quantities, in all the other equations in which it occurs.

The new equations thus formed, together with those in which this unknown quantity does not occur, are one less in number than the given equations, and contain one unknown quantity less, and may, by a succession of similar eliminations be still farther reduced in number and in the number of their unknown quantities, until only one equation is finally obtained; and the solution of all the given equations is thus reduced to that of one equation.

143. *Corollary.* When there are just as many equations as unknown quantities, the final equation of the preceding solution will, in general, contain but one unknown quantity, the value of which may be thence obtained; and this value, being substituted in the values of the other quantities, will lead to the determination of the values of all the unknown quantities.

144. *Corollary.* When the number of unknown quantities is more than that of the given equations, the final equation will contain several unknown quantities, and will therefore be indeterminate; so that *a problem is indeterminate, which gives fewer equations than unknown quantities.*

145. *Corollary.* When the number of unknown quantities is less than that of the given equations, only as many of the given equations are required to determine the values of the unknown quantities as there are unknown quantities; and the problem is therefore impossible, when the values of the un-

known quantities determined from the requisite equations do not satisfy the remaining equations.

146. *Problem. To solve two equations of the first degree with two unknown quantities.*

Solution. Suppose, as in art. 140, the given equations to be reduced to the forms

$$A x + B y + M = 0,$$
$$A' x + B' y + M' = 0:$$

in which x and y are the unknown quantities.

The value of x, obtained from the first of these equations, is

$$x = \frac{-B y - M}{A};$$

which, substituted in the second equation, gives

$$\frac{-A' B y - A' M}{A} + B' y + M' = 0.$$

The value of y is found from this equation, by art. 121, to be

$$y = \frac{A' M - A M'}{A B' - A' B};$$

which, substituted in the above value of x, gives

$$x = \frac{B M' - B' M}{A B' - A' B}.$$

147. *Corollary.* The value of x, obtained by the preceding solution would be zero, if its numerator were zero, that is, if

$$B M' = B' M.$$

But, in this case, if the first of the given equations is multiplied by B', and the second by B, these products become, by transposition and substitution,

$$A\,B'x = -B\,B'y - B'M,$$
$$A'B\,x = -B\,B'y - B\,M' = -B\,B'y - B'M;$$

whence

$$A\,B'x = A'\,Bx;$$

that is, the given equations involve the condition that two different multiples of x are equal. But this is impossible, unless

$$x = 0.$$

The value of y would, likewise, be zero, if we had

$$A'\,M = A\,M',$$

which leads to conclusions with regard to y, similar to those just obtained with regard to x.

148. *Corollary.* The denominators of the values of both the unknown quantities would be zero, if we had

$$A\,B' = A'\,B.$$

But, in this case, if the first of the given equations is multiplied by B' and the second by B, these products become, by transposition and substitution,

$$A\,B'x + B\,B'y = -B'M,$$
$$A'\,Bx + B\,B'y = A\,B'x + B\,B'y = -B\,M';$$

whence, we must have

$$B'\,M = B\,M';$$

that is, they involve the impossibility that the two unequal quantities $B'M$ and $B\,M'$ are equal.

149. *Corollary.* Both the terms of the fractional value of x would be zero, if we had

$$B\,M' = B'\,M, \text{ and } A\,B' = A'\,B.$$

But, in this case, if the first of the given equations is mul-

tiplied by B' and the second by B, the products become, by substitution,

$$A\,B'\,x + B\,B'\,y + B'\,M = 0,$$
$$A\cdot B\,x + B\,B'\,y + B\,M' = A\,B'x + B\,B'\,y + B'M = 0;$$

that is, the two given equations are equivalent to but one, and are, as in art. 144, indeterminate.

The product of the two equations

$$B\,M' = B'\,M, \text{ and } A\,B' = A'\,B,$$

is

$$A\,B\,B'\,M' = A'\,B\,B'\,M,$$

which, divided by $B\,B'$, is

$$A\,M' = A'\,M,$$

so that both the terms of the value of y would also be zero.

150. EXAMPLES.

1. Solve the two equations

$$3\,x + 2\,y = 118,$$
$$x + 5\,y = 191.$$

Ans. $x = 16, y = 35$.

2. Solve the two equations

$$\frac{x}{2} + \frac{y}{3} = 8,$$
$$\frac{x}{3} - \frac{y}{2} = 1.$$

Ans. $x = 12, y = 6$.

3. Solve the two equations

$$\frac{7+x}{5} - \frac{2\,x - y}{4} = 3\,y - 5,$$
$$\frac{5\,y - 7}{2} + \frac{4\,x - 3}{6} = 18 - 5\,x.$$

Ans. $x = 3, y = 2$.

4. Solve the two equations

$$a x = b y,$$
$$x + y = c.$$

$$Ans.\ x = \frac{b c}{a + b},\ y = \frac{a c}{a + b}.$$

5. A says to B, "give me \$100, and I shall have as much as you." "No," says B to A, "give me rather \$100, and then I shall have twice as much as you." How many dollars has each? *Ans.* A \$500, and B \$700.

6. Said a lad to his father, "How old are we?" "Six years ago," answered the latter, "I was one third more than three times as old as you; but three years hence, I shall be obliged to multiply your age by $2\frac{1}{8}$ in order to obtain my own." What is the age of each?

Ans. The father 36, the son 15 years.

7. A cistern containing 210 buckets, may be filled by 2 pipes. By an experiment, in which the first was open 4, and the second 5 hours, 90 buckets of water were obtained. By another experiment, when the first was open 7, and the other $3\frac{1}{2}$ hours, 126 buckets were obtained. How many buckets does each pipe discharge in an hour?

Ans. The first pipe discharges 15, and the second pipe discharges 6 buckets.

8. There is a fraction such, that if 1 be added to its numerator its value becomes $= \frac{1}{3}$; and if 1 be added to its denominator its value becomes $= \frac{1}{4}$. What fraction is it?

Ans. $\frac{4}{15}$.

9. Required to find two numbers such, that if the first be increased by a, and the second by b, the product of these two sums exceeds the product of the two numbers themselves by c; if, on the other hand, the first be increased by

a', and the second by b', the product of these sums exceeds the products of the two numbers themselves by c'.

Ans. The first is $\frac{a'c-ac'+aa'b'-aa'b}{a'b-ab'}$, the second is $\frac{bc'-b'c+abb'-a'bb'}{a'b-ab'}$.

10. A person had two barrels, and a certain quantity of wine in each. In order to obtain an equal quantity in each, he poured out as much of the first cask into the second, as the second already contained; then, again, he poured out as much of the second into the first as the first then contained, and lastly, he poured out again as much from the first into the second as the second still contained. At last he had 16 gallons of wine in each cask. How many gallons did they contain originally?

Ans. The first 22, the second 10 gallons.

11. 21 lbs. of silver lose 2 lbs. in water, and 9 lbs. of copper lose 1 lb. in water. Now, if a composition of silver and copper weighing 148 lbs. loses $14\frac{2}{3}$ lbs. in water, how many lbs. does it contain of each metal?

Ans. 112 lbs. of silver, and 36 lbs. of copper.

12. A given piece of metal, which weighs p lbs., loses e lbs. in water. This piece, however, is composed of two other metals A and B such, that p lbs. of A lose a lbs. in water, and p lbs. of B lose b lbs. How much does this piece contain of each metal?

Ans. $\frac{(b-e)p}{b-a}$ lbs. of A, and $\frac{(e-a)p}{b-a}$ lbs. of B.

13. According to Vitruvius, the crown of Hiero, king of Syracuse, weighed 20 lbs., and lost $1\frac{1}{4}$ lbs. in water. Assuming that it consists of gold and silver only, and that 19,64 lbs. of gold lose 1 lb. in water, and 10,5 lbs. of silver lose 1 lb. in

water. How much gold, and how much silver, did this crown contain?

Ans. 14,77 ... lbs. of gold, and 5,22 ... lbs. of silver.

151. *Problem. To solve any number of equations of the first degree with the same number of unknown quantities.*

Solution. Let there be *three* equations with *three* unknown quantities; these equations may, by art. 140, be reduced to the forms

$$A\,x + B\,y + C\,z + M = 0,$$
$$A'\,x + B'\,y + C'\,z + M' = 0,$$
$$A''\,x + B''\,y + C''\,z + M'' = 0.$$

The value of x, given by the first of these equations, is

$$x = \frac{-B\,y - C\,z - M}{A};$$

which, being substituted in the other two equations, and the resulting equations being reduced, as in art. 140, gives

$$(A\,B' - A'\,B)\,y + (A\,C' - A'\,C)\,z + A\,M' - A'\,M = 0,$$
$$(A\,B'' - A''B)\,y + (A\,C'' - A''C)\,z + A\,M'' - A''M = 0.$$

These equations, being solved, as in art. 146, give

$$y = \frac{(A'C'' - A''C')M + (A''C - A\,C'')M' + (A\,C' - A'C)M''}{(A'B'' - A''B')C + (A''B - AB'')C' + (AB' - A'B)C''},$$

$$z = \frac{(A''B' - A'B'')M + (AB'' - A''B)M' + (A'B - AB')M''}{(A'B'' - A''B')C + (A''B - AB'')C' + (AB' - A'B)C''};$$

in which the terms are arranged in groups in order to display the symmetry of the result; and these values, being substituted in the value of x, give

$$x = \frac{(B''C' - B'C'')M + (BC'' - B''C)M' + (B'C - BC')M''}{(A'B'' - A''B')C + (A''B - AB'')C' + (AB' - A'B)C''}$$

Examples to be solved by Elimination by Substitution.

If this method of solution be applied to a greater number of equations, it will lead to similar results.

152. EXAMPLES.

1. Solve the three equations

$$\begin{aligned} x + y + z &= 6, \\ 2x + 3y + 4z &= 20, \\ 3x + 7y + 5z &= 32. \end{aligned}$$

Ans. $x = 1,\ y = 2,\ z = 3$

2. Solve the three equations

$$\begin{aligned} y + \tfrac{1}{2}x &= 41, \\ x + \tfrac{1}{4}z &= 20\tfrac{1}{2}, \\ y + \tfrac{1}{5}z &= 34. \end{aligned}$$

Ans. $x = 18,\ y = 32,\ z = 10$

3. Solve the three equations

$$\begin{aligned} 53 - \tfrac{1}{2}x - \tfrac{1}{2}z &= y - 109, \\ \tfrac{1}{4}x + \tfrac{1}{8}y &= 26, \\ 5y &= 4z. \end{aligned}$$

Ans. $x = 64,\ y = 80,\ z = 100$

4. Solve the four equations

$$\begin{aligned} x + y + z + u &= 1, \\ 16x + 8y + 4z + 2u &= 9, \\ 81x + 27y + 9z + 3u &= 36, \\ 256x + 64y + 16z + 4u &= 100. \end{aligned}$$

Ans. $x = \tfrac{1}{4},\ y = \tfrac{1}{2},\ z = \tfrac{1}{4},\ u = 0.$

5. The sums of three numbers, taken two and two, are a, b, c. What are they?

Ans. $\tfrac{1}{2}(a + b - c),\ \tfrac{1}{2}(a + c - b),\ \tfrac{1}{2}(b + c - a).$

6. A, B, C compare their fortunes. A says to B, "give me \$700 of your money, and I shall have twice as much

Examples to be solved by Elimination by Substitution.

as you retain;" B says to C, "give me \$ 1400, and I shall have thrice as much as you have remaining;" C says to A, "give me \$ 420, and then I shall have 5 times as much as you retain." How much has each?

Ans. A \$ 980, B \$ 1540, C \$ 2380.

7. Three soldiers, in a battle, make \$ 96 booty, which they wish to share equally. In order to do this, A, who made most, gives B and C as much as they already had; in the same manner, B then divided with A and C; and after this, C with A and B. If, by these means, the intended equal division is effected, how much booty did each soldier make? *Ans.* A \$ 52, B \$ 28, C \$ 16.

8 A, B, C, D, E play together on this condition, that he who loses shall give to all the rest as much as they already have. First A loses, then B, then C, then D, and at last also E. All lose in turn, and yet at the end of the 5th game they all have the same sum, viz. each \$ 32. How much had each when they began to play?

Ans. A \$ 81, B \$ 41, C \$ 21, D \$ 11, E \$ 6.

153. *Second Method of solving the Problem of art.* 142, *called that of Elimination by Comparison. Find the value of either of the unknown quantities in all the equations in which it is contained; place either of the values thus obtained equal to each of the others, and the equations thus formed will be one less in number than those from which they are obtained, and will contain one unknown quantity less. By continuing this process on these new equations, the number of equations will finally be reduced to one.*

154. EXAMPLES.

1. *To solve any two equations of the first degree with two unknown quantities.*

Solution. These equations may, as in art. 146, be reduced to the forms

$$A\,x + B\,y + M = 0,$$
$$A'x + B'y + M' = 0.$$

The values of x, obtained from these equations, are

$$x = \frac{-B\,y - M}{A},$$
$$x = \frac{-B'\,y - M'}{A'};$$

which, being placed equal to each other, give

$$\frac{-B\,y - M}{A} = \frac{-B'\,y - M'}{A'},$$

whence

$$y = \frac{A'\,M - A\,M'}{A\,B' - A'\,B},$$

and, therefore,

$$x = \frac{B\,M' - B'\,M}{A\,B' - A'\,B};$$

being the same values as those obtained in art. 146.

2. Solve the three equations

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{13}{12},$$
$$\frac{1}{x} + \frac{1}{y} - \frac{1}{z} = \frac{7}{12},$$
$$\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = \frac{5}{12}.$$

Examples to be solved by Elimination by Comparison.

Solution. The values of x, obtained from these equations, are

$$x = \frac{12\,y\,z}{13\,y\,z - 12\,z - 12\,y},$$

$$x = \frac{12\,y\,z}{7\,y\,z - 12\,z + 12\,y},$$

$$x = \frac{12\,y\,z}{5\,y\,z + 12\,z - 12\,y};$$

the first of which being placed equal to each of the others gives, by reduction,

$$z = 4,$$
$$y = 3;$$

whence we get, from either value of x, by substitution,

$$x = 2.$$

3. Solve the two equations

$$7\,y = 2\,x - 3\,y,$$
$$19\,x = 60\,y + 621\tfrac{1}{4}.$$

Ans. $x = 88\tfrac{3}{4}$, $y = 17\tfrac{3}{4}$.

4. Solve the three equations

$$3\,x + 5\,y = 161,$$
$$7\,x + 2\,z = 209,$$
$$2\,y + z = 89.$$

Ans. $x = 17$, $y = 22$, $z = 45$.

5. Solve the three equations

$$\frac{1}{x} + \frac{1}{y} = a,$$

$$\frac{1}{x} + \frac{1}{z} = b,$$

$$\frac{1}{y} + \frac{1}{z} = c,$$

Ans. $x = \dfrac{2}{a + b - c}$, $y = \dfrac{2}{a - b + c}$, $z = \dfrac{2}{b + c - a}$.

Examples to be solved by Elimination by Comparison.

6. Solve the three equations

$$\frac{2}{x} - \frac{3}{5y} + \frac{1}{z} = 3\,\frac{4}{15},$$

$$\frac{1}{4x} + \frac{1}{y} + \frac{2}{z} = 6\,\frac{11}{72},$$

$$\frac{5}{6x} - \frac{1}{y} + \frac{4}{z} = 12\,\frac{1}{36}.$$

Ans. $x = 6,\ y = 9,\ z = \frac{1}{3}$.

7 A person has two horses, and two saddles, one of which cost \$ 50, the other \$ 2. If he places the best saddle upon the first horse, and the worst upon the second, then the latter is worth \$ 8 less than the other; but if he puts the worst saddle upon the first horse, and the best upon the other, then the latter is worth $3\frac{3}{4}$ times as much as the first. What is the value of each horse?

Ans. The first \$ 30, the second \$ 70.

8. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{8}$? *Ans.* $\frac{4}{5}$.

9. A wine merchant has two kinds of wine. If he mix 3 gallons of the worst with 5 of the best, the mixture is worth \$ 1 per gallon; but, if he mix $3\frac{1}{4}$ gallons of the worst with $8\frac{3}{4}$ gallons of the best, the mixture is worth \$ $1{,}03\frac{1}{3}$ per gallon. What does each wine cost per gallon?

Ans. The best \$ 1,12, the worst \$ 0,80.

10. A wine merchant has two kinds of wine. If he mix a gallons of the first with b gallons of the second, the mixture is worth c dollars per gallon; but, if he mix a' gallons of the first with b' gallons of the second, the mixture is worth c' dollars per gallon. What does each wine cost per gallon?

Examples to be solved by Elimination by Comparison.

Ans. The first $\frac{(a+b)\,b'c-(a'+b')\,bc'}{ab'-a'b}$ dollars, the second $\frac{(a+b)\,a'c-(a'+b')\,ac'}{a'b-ab'}$ dollars.

11. Three masons, A, B, C, are to build a wall. A and B, jointly, could build this wall in 12 days; B and C could accomplish it in 20 days; A and C would do it in 15 days. What time would each take to do it alone?

Ans. A requires 20, B 30, C 60 days.

12. Three laborers are employed in a certain work. A and B would, together, complete it in a days; A and C require b days; B and C require c days. In what time would each accomplish it singly?

Ans. A in $\frac{2abc}{bc+ac-ab}$ days, B in $\frac{2abc}{bc+ab-ac}$ days, C in $\frac{2abc}{ab+ac-bc}$ days.

13. A cistern may be filled by three pipes, A, B, C. By the pipes A and B, it could be filled in 70 minutes; by the pipes A and C, in 84 minutes; and by the pipes B and C, in 140 minutes. In what time would each pipe fill it?

Ans. A in 105, B in 210, C in 420 minutes.

14. A, B, C play faro. In the first game A has the bank, B and C stake the third part of their money, and win. In the second game B has the bank, A and C stake the third part of their money and also win. Then C takes the bank, A and B stake the third part of their money and also win. After this third game they count their money, and find that they have all the same sum of 64 ducats. How much had each when they began to play?

Ans. A had 75, B 63, C 54.

Elimination by the method of the Greatest Common Divisor.

15. Five friends, A, B, C, D, E, jointly spend \$ 879 at an inn. This sum is to be paid by one of them; but, on consultation, they find that none of them had, alone, enough for this purpose. If, then, one of them is to pay it, the others must give him a part of their money. A can pay, if he receives one fourth; B, if he receives one fifth; C, if he receives one sixth; D, if he receives one seventh; and E, if he receives one eighth of the others' money. How much has each?

***Ans.* A \$ 319, B \$ 459, C \$ 543, D \$ 599, E \$ 639.**

155. *Third Method of solving the Problem of art, 142, called that of Elimination by the method of the greatest Common Divisor.*

Solution. This method is generally inapplicable to transcendental equations, but can be successfully applied in all other cases to eliminate one unknown quantity after another, until the given equations are reduced to one.

In order to eliminate an unknown quantity from two equations which contain it, reduce them as in arts.* 105 *and* 118, *and arrange their terms according to the powers of the quantity to be eliminated; taking out each power as a factor from the terms which contain it.

It being now recollected that the second member of each of these equations is zero, it will appear evident that, if the first members are divided one by the other, the remainder arising from this division must likewise be equal to zero; for this remainder is the difference between the dividend and a certain multiple of the divisor, that is, between zero and a certain multiple of zero.

Hence, divide one of these first members by the other, and proceed, as in arts. 60, &c., to find their greatest common divisor; each successive remainder may be placed equal to zero. But a remainder will at last be obtained, which does not contain the quantity to be eliminated; and the equation, formed from placing this remainder equal to zero, is the equation from which this quantity is eliminated.

By eliminating, in this way, the unknown quantity from either of the equations which contain it, taken with each of the others, a number of equations is formed one less than that of the given equations, and containing one less unknown quantity; and to which this process of elimination may be again applied until one equation is finally obtained.

156. *Scholium.* It sometimes happens, that the first members have a common divisor which contain the given unknown quantity; and, in this case, the process cannot be continued beyond this divisor.

But as the given first members are multiples of their common divisor, they must be rendered equal to zero by those values of the unknown quantities which render the common divisor equal to zero; that is, the two given equations are satisfied by such values of the unknown quantities; so that, though they are in appearance distinct equations, they are, in reality, equivalent to but one equation, that is, to the equation formed by placing their common divisor equal to zero.

157. *Scholium.* Care must be taken that no factor be suppressed which may be equal to zero.

158. EXAMPLES.

1. Obtain one equation with one unknown quantity from the two equations

$$x^3 + y\,x^2 - y^3 + 5 = 0,$$
$$x^3 + y^2\,x - 5 = 0,$$

by the elimination of x.

Solution. Divide the first members as follows.

$$\begin{array}{l|l} x^3 + y\,x^2 - y^3 + 5 & x^3 + y^2\,x - 5 \\ \cline{2-2} \underline{x^3 + y^2\,x - 5} & 1 \end{array}$$

1st Rem. $y\,x^2 - y^2\,x - y^3 + 10$.

Divide the preceding divisor by this remainder after multiplying by y to render the first term divisible.

$$\begin{array}{l|l} y\,x^3 + y^4\,x - 5\,y & y\,x^2 - y^2\,x - y^3 + 10 \\ \cline{2-2} \underline{y\,x^3 - y^2\,x^2 - y^3\,x + 10\,x} & x + y \\ y^2\,x^2 + (2\,y^3 - 10)\,x - 5\,y & \\ \underline{y^2\,x^2 - \quad y^3 \qquad x - y^4 + 10\,y} & \end{array}$$

2d Rem. $(3\,y^3 - 10)\,x + y^4 - 15\,y$.

Divide the preceding divisor by this remainder after multiplying by $(3\,y^3 - 10)$ to render the first term divisible.

$$\begin{array}{l} y\,x^2 - y^2\,x - y^3 + 10 \\ \underline{3\,y^3 - 10} \end{array}$$

$$\begin{array}{r|r|l||r|l} 3\,y^3 & y\,x^2 - \;3\,y^5 & x - \;3\,y^6 - 100 & 3\,y^3 & x + \;\;y^4 \\ -10 & +10\,y^2 & \;\;+40\,y^3 & -10 & \;-15\,y \\ \cline{4-5} 3\,y^3 & y\,x^2 + \;\;y^5 & x & \multicolumn{2}{l}{y\,x, \quad -\;4\,y^5} \\ -10 & -15\,y^2 & & \multicolumn{2}{r}{+25\,y^2} \\ \hline & -\;4\,y^5 & x - \;3\,y^6 - 100 & \multicolumn{2}{l}{\text{Multiply by}} \\ & +25\,y^2 & \;\;+40\,y^3 & \multicolumn{2}{l}{(3\,y^3 - 10),} \\ \multicolumn{3}{c}{3\,y^3 - 10} & & \end{array}$$

$$\begin{array}{r} \hline (3\,y^3-10)\,(-4\,y^5+25\,y^2)\,x - 9\,y^9 + 150\,y^6 - 700\,y^3 + 1000 \\ (3\,y^3-10)\,(-4\,y^5+25\,y^2)\,x - 4\,y^9 + \;\,85\,y^6 - 375\,y^3 \qquad\quad \\ \hline -5\,y^9 + 65\,y^6 - 325\,y^3 + 1000, \end{array}$$

whence the required equation is

$$-5\,y^9 + 65\,y^6 - 325\,y^3 + 1000 = 0.$$

2. Obtain one equation with one unknown quantity from the two equations

$$x^3 + y^3 = a,$$
$$x^5 + y^5 = b,$$

by the elimination of x.

Ans. $(y^3 - a)^5 - (y^5 - b)^3 = 0.$

3. Obtain one equation with one unknown quantity from the two equations

$$x^2 + y^2 = 2,$$
$$x^4 + x^3 y + x^2 y^2 + x y^3 + y^4 = 1,$$

by the elimination of x.

Ans. $y^8 - 4 y^6 + 14 y^4 - 20 y^2 + 9 = 0.$

4. Obtain one equation with one unknown quantity from the two equations

$$x^2 + x y + y^2 = 1,$$
$$x^3 + y^3 = 0,$$

by the elimination of x.

Ans. $4 y^6 - 6 y^4 + 3 y^2 - 1 = 0.$

5. Obtain one equation with one unknown quantity from the two equations

$$x^3 + y x^2 + x + y = 4,$$
$$x^3 + x^2 + y x = 3,$$

by the elimination of x.

Ans. Either $y - 1 = 0$, or $y^2 - 3 y + 21 = 0.$

6. Obtain one equation with one unknown quantity from the three equations

$$x + y + z = a,$$
$$x z + x y + y z = b,$$
$$x y z = c,$$

by the elimination of x and y.

Ans. $z^3 - a z^2 + b z - c = 0.$

Examples of Elimination by the method of the Greatest Common Divisor.

7. Obtain one equation with one unknown quantity from the three equations

$$x + y + z = a,$$
$$x^2 + y^2 + z^2 = b,$$
$$x y + x z + y z = c,$$

by the elimination of x and y.

Ans. These three equations involve an impossibility unless

$$a^2 - b - 2 c = 0;$$

and in case this equation is satisfied by the given values of a, b, and c, the three given equations are equivalent to but two, one of them being superfluous, and, by the elimination of x, they give the indeterminate equation with two unknown quantities

$$y^2 + y z + z^2 - a y - a z + c = 0.$$

8. Obtain one equation with one unknown quantity from the three equations

$$x + y^2 = 4,$$
$$y + z^2 = 2,$$
$$x + x^2 = 10,$$

by the elimination of x and y.

Ans. $z^8 - 8 z^6 + 16 z^4 + z - 10 = 0.$

9. Obtain one equation with one unknown quantity from the four equations

$$x + y + z + u = a,$$
$$x y + x z + x u + y z + y u + z u = b,$$
$$x y z + x y u + x z u + y z u = c,$$
$$x y z u = e,$$

by the elimination of x, y, and z.

Ans. $u^4 - a u^3 + b u^2 - c u + e = 0.$

10. Solve the two equations

$$y\,x^3 - x^3 + x = 3,$$
$$y\,x\,(y\,x^2 + 1) - x^3 + x = 6.$$

Solution. The elimination of x gives

$$3\,y - 3 = 0, \text{ or } y = 1;$$

which, being substituted in the first of the given equations produces

$$x = 3.$$

11. Solve the two equations

$$x^2y^4 - 8\,y^2x^2 + 16\,x^2 = 90\,x\,y + 60\,(x - y^2) - 720\,(y - 1)$$
$$\frac{(y^2 - 4\,y + 4)\,x}{5} = 3 - \frac{12}{x}.$$

Ans. $x = 4,\ y = 2.$

12. Solve the three equations

$$x\,y + z = 5,$$
$$x\,y\,z + z^2 = 15,$$
$$x\,y^2 + x^2\,y - 2\,x + 2\,z = 8.$$

Ans. $x = 2,\ y = 1,\ z = 3.$

159. *Problem. To solve two equations of the first degree by Elimination by Addition and Subtraction.*

Solution. The given equations may, as in art. 146, be reduced to the forms

$$A\,x + B\,y + M = 0,$$
$$A'\,x + B'\,y + M' = 0.$$

The process of the preceding article, being applied to these equations in order to eliminate x, will be found to be the same as to

Multiply the first equation by A′ *the coefficient of* x *in the second, multiply the second by* A *the coefficient of* x *in the first, and subtract the first of these products from the second.*

Thus, these products are

$$A\,A'\,x + A'\,B\,y + A'\,M = 0,$$
$$A\,A'\,x + A\,B'\,y + A\,M' = 0;$$

and the difference is

$$(A\,B' - A'\,B)\,y + A\,M' - A'\,M = 0;$$

whence

$$y = \frac{A'\,M - A\,M'}{A\,B' - A'\,B}.$$

In the same way y might have been eliminated by multiplying the first equation by B', and the second by B, and the difference of these products is

$$(A\,B' - A'\,B)\,x + B'\,M - B\,M' = 0;$$

whence

$$x = \frac{B\,M' - B'\,M}{A\,B' - A'\,B},$$

the values of x and y thus obtained being the same as those given in art. 146.

160. *Corollary.* This process may be applied with the same facility to any equations of the first degree.

161. EXAMPLES.

1. Solve, by the preceding process, the two equations

$$13\,x + 7\,y - 341 = 7\tfrac{1}{2}\,y + 43\tfrac{1}{2}\,x,$$
$$2\,x + \tfrac{1}{2}\,y = 1.$$

Ans. $x = -12,\ y = 50.$

2. Solve, by the preceding process, the two equations

$$\tfrac{1}{6}\,x + \tfrac{1}{4}\,y = 6,$$
$$\tfrac{1}{4}\,x + \tfrac{1}{6}\,y = 5\tfrac{2}{3}.$$

Ans. $x = 12,\ y = 16.$

Examples of Elimination by Addition and Subtraction.

3. Solve, by the preceding process, the three equations

$$x + y + z = 30,$$
$$8x + 4y + 2z = 50,$$
$$27x + 9y + 3z = 64.$$

Ans. $x = \frac{2}{3},\ y = -7,\ z = 36\frac{1}{3}$.

4. Solve, by the preceding process, the three equations

$$3x - 100 = 5y + 360,$$
$$2\tfrac{1}{3}x + 200 = 16\tfrac{1}{2}z - 610,$$
$$2y + 3z = 548.$$

Ans. $x = 360,\ y = 124,\ z = 100$.

5. Solve, by the preceding process, the four equations

$$x - 9y + 3z - 10u = 21,$$
$$2x + 7y - z - u = 683,$$
$$3x + y + 5z + 2u = 195,$$
$$4x - 6y - 2z - 9u = 516.$$

Ans. $x = 100,\ y = 60,\ z = -13,\ u = -50$.

6. Solve, by the preceding process, the four equations

$$\tfrac{1}{3}x + \tfrac{1}{5}y + \tfrac{2}{7}z = 58,$$
$$\tfrac{5}{4}x + \tfrac{1}{6}y + \tfrac{1}{3}z = 76,$$
$$\tfrac{1}{2}x + \tfrac{3}{8}z + \tfrac{1}{5}u = 79,$$
$$y + z + u = 248.$$

Ans. $x = 12,\ y = 30,\ z = 168,\ u = 50$.

7. Solve, by the preceding process, the six equations

$$x + y + z + t + u = 20,$$
$$x + y + z + u + w = 21,$$
$$x + y + z + t + w = 22,$$
$$x + y + u + t + w = 23,$$
$$x + z + u + t + w = 24,$$
$$y + z + u + t + w = 25.$$

Ans. $x = 2,\ y = 3,\ z = 4,\ u = 5,\ t = 6,\ w = 7$.

Examples of Elimination by Addition and Subtraction.

8. A person has two large pieces of iron whose weight is required. It is known that $\frac{2}{5}$ths of the first piece weigh 96 lbs. less than $\frac{3}{4}$ths of the other piece; and that $\frac{5}{8}$ths of the other piece weigh exactly as much as $\frac{4}{9}$ths of the first. How much did each of these pieces weigh?

Ans. The first weighed 720, and the second 512 lbs.

9. \$2652 are to be divided amongst three regiments, in such a way, that each man of that regiment which fought best, shall receive \$1, and the remainder is to be divided equally among the men of the other two regiments. Were the dollar adjudged to each man in the first regiment, then each man of the two remaining regiments would receive \$$\frac{1}{2}$; if the dollar were adjudged to the second regiment, then each man of the other two regiments would receive \$$\frac{1}{3}$; finally, if the dollar were adjudged to the third regiment, each man of the other two regiments would receive \$$\frac{1}{4}$. What is the number of men in each regiment?

Ans. 780 men in the first, 1716 in the second, and 2028 in the third regiment.

10. To find three numbers such that if 6 be added to the first and second, the sums are to one another as 2 : 3; if 5 be added to the first and third, the sums are as 7 : 11; but if 36 be subtracted from the second and third, the remainders are as 6 : 7.

Ans. 30, 48, 50.

CHAPTER IV.

NUMERICAL EQUATIONS.

SECTION I.

Indeterminate Coefficients.

162. ***Theorem.*** *If a polynomial*

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$$

is such, as to be equal to zero independently of x, *that is, if it is equal to zero whatever values are given to* x, *it must always be the case that*

$$A = 0,\ B = 0,\ C = 0,\ D = 0,\ E = 0,\ \&c.;$$

that is, that the aggregate of all the coefficients of each power of x *is equal to zero, and also the aggregate of all the terms which do not contain* x *is equal to zero.*

Proof. Since the equation

$$A + Bx + Cx^2 + Dx^3 + \&c. = 0$$

is true for every value which can be given to x, it must be true when we make

$$x = 0;$$

in which case all the terms of the first member vanish ex cept the first, and we have

$$A = 0.$$

Indeterminate Coefficients.

This equation, being subtracted from the given equation, gives

$$Bx + Cx^2 + Dx^3 + \&c. = 0;$$

and, dividing by x,

$$B + Cx + Dx^2 + \&c. = 0;$$

whence we may prove as above, that

$$B = 0.$$

By continuing this process, we can prove that

$$C = 0,\ D = 0,\ E = 0,\ \&c.$$

163. *Theorem. If two polynomials*

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.,$$
$$A' + B'x + C'x^2 + D'x^3 + E'x^4 + \&c.$$

are identical, that is, equal, independently of x, *must always be the case that*

$$A = A',\ B = B',\ C = C',\ D = D',\ \&c.$$

Proof. For the equation

$$A + Bx + Cx^2 + \&c. = A' + B'x + C'x^2 + \&c.$$

gives, by transposition,

$$(A - A') + (B - B')x + (C - C')x^2 + \&c. = 0;$$

whence, by the preceding theorem,

$$A - A' = 0,\ B - B' = 0,\ C - C' = 0,\ \&c.;$$

that is,

$$A = A',\ B = B',\ C = C',\ \&c.$$

SECTION II.

Derivation.

164. *Definition.* When quantities are so connected that their values are dependent upon each other, each is said to be a *function* of the others: which are called *variables* when they are supposed to be changeable in their values, and *constants* when they are supposed to be unchangeable.

Thus if $$y = a x + b$$
y is a function of the a, b, and x; but if x is variable while a and b are constant, it is more usual to regard y as simply a function of x.

165. *Definition.* In the case of a change in the value of a function, arising from an infinitely small change in the value of one of its variables, *the relative rate of change of the function and the variable,* that is, the ratio of the change in the value of the function to that in the value of the variable, is called *the derivative of the function.*

The derivative of the derivative of a function is called *the second derivative of the function;* the derivative of the second derivative is called *the third derivative;* and so on.

166. *Corollary. The derivative of a constant is zero.*

167. *Corollary. The derivative of the variable, regarded as a function of itself, is unity; and the second derivative is zero.*

The Derivative of the sum of any Functions.

168. *Theorem. The derivative of the sum of two functions is the sum of their derivatives.*

Proof. Let the two functions be u and v, and let their values, arising from an infinitesimal change i in the value of their variable, be u' and v'; the increase of their sum will be

$$(u'+v')-(u+v)$$

or

$$u'-u+v'-v,$$

and therefore the derivative of the sum is

$$\frac{u'-u}{i}+\frac{v'-v}{i},$$

which is obviously the sum of their derivatives.

169. *Corollary.* By reversing the sign of v, it may be shown, in the same way, that *the derivative of the difference of two functions is the difference of their derivatives.*

170. *Corollary. The derivative of the algebraic sum of several functions connected by the signs + and — is the algebraic sum of their derivatives.*

171. *Corollary.* If, in this sum, any function is repeated any number of times, its derivative must be repeated the same number of times; in other words, *if a function is multiplied by a constant its derivative must be multiplied by the same constant.*

Thus, if the derivatives of u, v, and w are respectively U, V, and W, and if a, b, c, and e are constant, the derivative of

$$a\,u+b\,v-c\,w+e$$

is

$$a\,U+b\,V-c\,W.$$

172. ***Problem.*** *To find the derivative of any power of a variable.*

Solution. Let the variable be a and the power a^n, and let b differ infinitely little from a; the derivative of a^n is then

$$\frac{b^n - a^n}{b - a}.$$

Now when b is equal to a, the value of this quotient is, by art. 51,
$$n\,a^{n-1};$$
and this must differ from the present value of this quotient, by an infinitely small quantity, which being neglected gives
$$n\,a^{n-1}$$
for the derivative of a^n.

The derivative of any power of a variable is, therefore, found by multiplying by the exponent, and diminishing the exponent by unity.

173. *Corollary.* The derivative of $m\,a^n$ when m is constant and a variable is $n\,m\,a^{n-1}$.

174. ***Problem.*** *To find the derivative of any power of a function.*

Solution. Let the variable be a, the function u, and the power u^n; let b differ infinitely little from a, and let v be the corresponding value of u; if U is the derivative of u and U' that of u^n, we have

$$U' = \frac{v^n - u^n}{b - a}, \text{ and } U = \frac{v - u}{b - a}.$$

But, by art. 51,

$$\frac{v^n - u^n}{v - u} = n\,u^{n-1},$$

which multiplied by

$$U = \frac{v - u}{b - a},$$

gives

$$U' = \frac{v^n - u^n}{v - u} \cdot \frac{v - u}{b - a} = \frac{v^n - u^n}{b - a} = n\, u^{n-1}\, U.$$

The derivative of any power of a function is, therefore, found by multiplying by the exponent and by the derivative of the function, and diminishing the exponent by unity.

175. EXAMPLES.

Find the derivatives of the following functions in which x is the variable.

1. x^2. *Ans.* $2\,x$.
2. x^3. *Ans.* $3\,x^2$.
3. $x^n + a\,x^m + b\,x^p + \&c.$
   Ans. $n\,x^{n-1} + m\,a\,x^{m-1} + p\,b\,x^{p-1} + \&c.$
4. $A + B\,x + C\,x^2 + D\,x^3 + E\,x^4 + F\,x^5 + \&c.$
   Ans. $B + 2\,C\,x + 3\,D\,x^2 + 4\,E\,x^3 + 5\,F\,x^4 + \&c.$
5. $a + x$. *Ans.* 1.
6. $(a + x)^2$. *Ans.* $2\,(a + x)$.
7. $(a + x)^3$. *Ans.* $3\,(a + x)^2$.
8. $(a + x)^n$. *Ans.* $n\,(a + x)^{n-1}$.
9. $(a + b\,x)^2$. *Ans.* $2\,b\,(a + b\,x)$.
10. $(a + b\,x)^n$. *Ans.* $n\,b\,(a + b\,x)^{n-1}$.

176. ***Problem. To find the derivative of the product of two functions.***

Solution. **Let u and v be the functions, and U and V their derivatives; then, since the derivative is the rate of change of the function to that of the variable, it is evident**

that when the variable is increased by the infinitesimal i, that the functions will become

$$u + U i \text{ and } v + V i.$$

The product will therefore change from

$$u v$$

to

$$(u + U i)(v + V i) = u v + v U i + u V i + U V i^2,$$

and the increase of the product is

$$v U i + u V i + U V i^2;$$

the ratio of which to i is

$$v U + u V + U V i,$$

or, neglecting the infinitesimal $U V i$, it is

$$v U + u V;$$

that is, *the derivative of a product of two functions is equal to the sum of the two products obtained by multiplying each function by the derivative of the other function.*

177. *Corollary.* The derivative of

$$(x - a)^n v$$

is, then,

$$n (x - a)^{n-1} v + (x - a)^n V,$$

because the derivative of

$$(x - a)^n$$

is

$$n (x - a)^{n-1}.$$

SECTION III.

Numerical Equations.

178. *Definition.* A *numerical equation* is one all whose coefficients are given in numbers, so that it involves no literal expressions except those denoting the unknown quantities.

179. ***Problem.*** ***To solve a numerical equation.***

Solution. Let the equation be reduced as in arts. 105 and 118, *to the form*

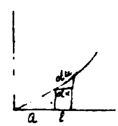
$$u = 0.$$

Find by trial a value of the unknown quantity x *which nearly satisfies this equation, and let this value be* a*; substitute this value in the given equation, and let the corresponding value of* u *be* m. *A correction* e *in the value of* a *is then to be found, which shall reduce the value of* u *from* m *to zero. Now, if* U *is the derivative of* u, *and if* M *is the value of* U *which corresponds to*

$$x = a,$$

M is, by art. 165, the rate at which u changes in comparison with x, so that when

$$x = a + e$$
$$u = m + M\,e = 0,$$



and therefore

$$e = -\frac{m}{M},\; x = a + e = a - \frac{m}{M}.$$

By this means a value of x *is found which is not*

perfectly accurate, because M is not the rate at which u varies during the whole interval from

$$x = a \text{ to } x = a + e;$$

but only while x differs infinitely little from a.

Calling, therefore, a′ *this approximate value of* x, *we have*

$$a' = a - \frac{m}{M},$$

which may be used in the same way in which a *was, in order to find a new approximate value* a″ *of* x*; and if* m′ *and* M′ *denote the corresponding values of* u *and* U, *we shall have*

$$a'' = a' - \frac{m'}{M'}.$$

In the same way, may the approximation be continued to any degree of accuracy.

180. *Problem.* To determine the *rate of approximation* in the preceding solution.

Solution. This is a most important, practical point, and the determination of it will be found to facilitate the solution.

Now, it may be observed, that since e is the correction of a, its magnitude shows the degree of accuracy which belongs to a, and the accuracy of e is, obviously, the same with that of

$$a' = a + e.$$

The comparative accuracy of the approximate value of a, and the succeeding approximate value a', is, then, the same with the magnitude of e compared with the error of e.

Now, in determining e, M was supposed to be the rate at which u changed throughout the whole interval in the change of x from a to $a + e$. But if the rate of change of

M is denoted by N, that is, if N is the derivative of M, the value of M, at the end of this interval when x is $a+e$, must be increased to

$$M + N e.$$

In the middle of the interval when x is $a + \frac{1}{2} e$, the value of M is

$$M + \tfrac{1}{2} N e,$$

which may be regarded as the average value of the rate of its increase, throughout the interval. When x, therefore, becomes $a + e$, u becomes

$$m + (M + \tfrac{1}{2} N e) e = 0,$$

or

$$m + M e + \tfrac{1}{2} N e^2 = 0;$$

whence by transposition and division

$$e = \frac{-m}{M} - \frac{N}{2M} e^2;$$

which differs from

$$e = -\frac{m}{M}$$

by the term

$$-\frac{N}{2M} e^2,$$

which may, therefore, be regarded as the error of e; and its comparison with e gives the rate of approximation.

181. *Corollary. If the value of* a *is accurate to a given place of decimals, as the* g*th, this will be shown by the magnitude of* e, for we shall find

$$e < \frac{1}{10^g},$$

and, consequently,

$$e^2 < \frac{1}{10^{2g}}.$$

If also the value of $\frac{\mathrm{N}}{2\,\mathrm{M}}$ *is found to be such* that

$$\frac{N}{2\,M} < \frac{1}{10^k},$$

then the inaccuracy of e^2 or of a' is

$$\frac{N}{2\,M}\,e^2 < \frac{1}{10^{2g+k}},$$

that is, a' *is accurate to the* $(2\,g + k)$*th place of decimals and the division of* m *by* M *may be carried to this extent.*

182. *Corollary.* When the given equation has the form

$$u = h,$$

in which h is a given number, it may be brought to the form

$$u - h = 0,$$

so that the value of the final member when

$$x = a$$

is

$$m - h,$$

while the value of the derivative is M, because h does not vary, and, therefore,

$$e = -\frac{m - h}{M} = \frac{h - m}{M},$$

which is often a more convenient form in practice than that of art. 179.

183. EXAMPLES.

1. Solve the equation

$$x^3 - 3x = -1,$$

which has three roots, the first being nearly 2, the second nearly 0, and the third nearly -2.

Solution. This equation, compared with arts. 179 - 182, gives

$$u = x^3 - 3x, \; h = -1,$$

$$U = 3x^2 - 3; \text{ deriv. of } U = 6x.$$

Hence, if

$$a = 2,$$

$$m = 8 - 6 = 2, \; M = 12 - 3 = 9, \; N = 12,$$

$$\frac{N}{2M} = \frac{12}{18} < 1, \; k = 0,$$

$$e = \frac{h - m}{M} = \frac{-1 - 2}{9} = -\tfrac{1}{3} = -{\cdot}3, \; g = 0,$$

$$a' = a + e = 2 - 0{\cdot}3 = 1{\cdot}7,$$

$$h - m' = -{\cdot}813, \; M' = 5{\cdot}67,$$

$$e' = -{\cdot}15, \; g' = 0, \; a' = 1{\cdot}55,$$

$$h - m'' = -0{\cdot}073875, \; M'' = 4{\cdot}2075,$$

$$e'' = -{\cdot}018, \; g'' = 1, \; a'' = 1{\cdot}532,$$

$$h - m''' = 0{\cdot}000359232, \; M''' = 4{\cdot}041072,$$

$$e''' = 0{\cdot}00008889, \; g''' = 4, \; a''' = 1{\cdot}53208889,$$

which is accurate to $2g''' = 8$ places of decimals.

This process may be arranged in the following form, in the first column of which, h is placed at the top, and the successive values of $-m$ above each horizontal line with those of $h - m$ below it. In the second column are placed the successive values of the divisor M. In the third column the first approximation a is placed at the top of the table,

and the successive values of e, above each line with those of $a+e$ below it.

		$M.$		
$h.$	—1·		2·	$a.$
$-m.$	—2·			
$h-m.$	—3·	9·	—0·3	$e.$
	0·187		1·7	$a+e.$
	—0·813	5·67	—0·15	
	0·926125		1·55	
	—0·073875	4·2075	—0·018	
	1·000359232		1·532	
	0·000359232	4·041072	0·00008889	
			1·53208889	

In the same way may the second and third roots be found, as follows.

When $x=0·3,\ \dfrac{N}{2M}=\dfrac{180}{546},\ k=0.$

—1·		0·
0·		
—1·	—3·	0·3
0·873		0·3
—0·127	—2·73	0·04
0·980696		0·34
0·019304	—2·6532	0·0073
—1·000009615183		0·3473
0·000009615183	—2·63814813	—0·0000036446
	The second root =	0·3472963554

When $x=-2,\ \dfrac{N}{2M}=-\dfrac{2}{3},\ k=0.$

—1·		—2·
2·		
1·	9·	0·1
1·159		—1·9
0·159	7·83	0·02
1·004672		—1·88
0·004672	7·6032	0·000614
	The third root =	—1·879386

Solution of Numerical Equations.

2. Solve the equation

$$x^3 - 12x = -132,$$

which has a root nearly equal to -6.

Ans. $-5{\cdot}87205266$.

3. Solve the equation

$$x^4 + 8x^2 + 16x = 440,$$

which has two roots, the first being nearly 4 and the second nearly -4. *Ans.* $3{\cdot}97601$ and $-4{\cdot}350577$.

4. Solve the equation

$$2x^4 - 20x = -19,$$

which has two roots, the first being nearly 1, and the second nearly 2. *Ans.* $1{\cdot}0928$ and $1{\cdot}59407$.

5. Solve the equation

$$5x^3 - 6x = -2,$$

which has three roots, the first being nearly 1, the second nearly 0, and the third nearly -1.

Ans. $0{\cdot}856$, $0{\cdot}3785$, $-1{\cdot}2345$.

184. *Problem. To find any root of a number.*

Solution. If the required root is the nth root of the number h, this problem is equivalent to solving the equation

$$x^n = h;$$

so that, if the preceding solution is applied to this case, we have

$$u = x^n,\ U = n\,x^{n-1}.$$

185. *Corollary.* When $x = a$,

$$m = a^n,\ M = n\,a^{n-1},\ N = n\,(n-1)\,a^{n-2}$$

$$\frac{N}{2M} = \frac{n\,(n-1)\,a^{n-2}}{2\,n\,a^{n-1}} = \frac{n-1}{2a}.$$

186. *Corollary.* It may be observed, since

$$(10^b\,e)^n = 10^{nb}\,e^n;$$

so that if,

$$e < 10,$$

$$10^b e < 10^{b+1}, (10^b e)^n < 10^{n(b+1)};$$

and if

$$e > 1,$$

$$10^b e > 10^b, (10^b e)^n > 10^{nb};$$

that is, if the root is between 10^b and 10^{b+1} the nth power is between 10^{nb} and 10^{nb+n}; or, otherwise, if the left hand significant figure of the root is b places from the decimal point, that of the power must be as many as b times n places from this point, and less than $b+1$ times n places from it; which, combined with the preceding articles, gives the following rule for finding the root of a number.

Divide the given number into portions or periods beginning with the decimal point, and let each portion or period contain the number of places denoted by the exponent of the power.

Find the greatest integral power contained in the left hand period; and the root of this power is the left hand figure of the required root, and is just as many places distant from the decimal point as the corresponding period is removed by periods from this point.

Raise the approximate root thus found to the given power and subtract it from the given number, and leave the remainder as a dividend.

Raise, again, this approximate root to the power next inferior to the given power, and multiply it by the exponent of the given power for a divisor.

The quotient of the dividend by the divisor gives the next figure or figures of the root.

The new approximate root, thus found, is to be used in the same way for a new approximation.

The number of places to which each division may usually be carried, is so far as to want but one place of doubling the number of places, to which the preceding approximation was found to be accurate.

186. EXAMPLES.

1. Find the fourth root of

5327012345·641.

Solution. In the following solution, the columns are the same as the first and second columns in art. 183, except that the top number of the second column is the root which is separated by space into the parts obtained by each successive division, and the number at the top of the first column is divided by spaces into periods.

53	2701	2345·	641	2 7* 0·16
16				
37	2701	2345·	641	32000000
53	1441			
	1260	2345·	641	78732000

Ans. 270·16.

2. Find the 4th root of 79502005521. *Ans.* 531.

3. Find the 3d root of 75686967. *Ans.* 423.

4. Find the 3d root of 128787625. *Ans.* 505.

5. Find the 3d root of 20548344701. *Ans.* 5901.

6. Find the 3d root of 512768384064. *Ans.* 8004.

7. Find the 3d root of 524581674·625. *Ans.* 806·5.

* This figure must, in the present case, be found by trial, because the first quotient is so inaccurate.

Roots of Fractions.

8. Find the 3d root of 1003·003001. *Ans.* 10·01.
9. Find the 3d root of 0·756058031. *Ans.* 0·911.
10. Find the 3d root of 0·000003442951. *Ans.* 0·0151.
11. Find the 5th root of 418227202051. *Ans.* 211.
12. Find the 4th root of 75450765·3376. *Ans.* 93·2.
13. Find the 5th root of 0·0000168505881551. *Ans.* 0·111.
14. Find the 4th root of 2526·88187761. *Ans.* 7·09.
15. Find the 3d root of 12. *Ans.* 2·289 +.
16. Find the 3d root of 28·25. *Ans.* 3·045 +.

187. *Corollary.* The roots of fractions can be found by reducing them to their lowest terms, and extracting the roots of their numerators and denominators separately.

The roots of mixed numbers can be found by reducing them to improper fractions.

188. EXAMPLES.

1. Find the 3d root of $\frac{27}{64}$. *Ans.* $\frac{3}{4}$.
2. Find the 3d root of $\frac{17576}{24389}$. *Ans.* $\frac{26}{29}$.
3. Find the 3d root of $\frac{375}{3993}$. *Ans.* $\frac{5}{11}$.
4. Find the 3d root of $6\frac{1275}{1331}$. *Ans.* $1\frac{10}{11}$.
5. Find the 4th root of $3\frac{13}{81}$. *Ans.* $1\frac{1}{3}$.

189. *Corollary.* In the case of the square root, we have

$$u = x^2,\ U = 2x,$$
$$m = a^2,\ M = 2a;$$

and, since the square of $a + h$ is

$$(a+h)^2 = a^2 + 2\,a\,h + h^2 = a^2 + (2\,a + h)\,h$$

it is unnecessary to find the square of the whole root at each successive approximation; for the square of a being already subtracted, it is sufficient to subtract $(2\,a + h)\,h$ from the remainder, in order to obtain the next remainder. In this way, we obtain the following rule for the extraction of the square root.

To extract the square root of a number, divide it into periods of two figures each, beginning with the place of units.

Find the greatest square contained in the left hand period, and its root is the left hand figure of the required root.

Subtract the square of the root thus found from the left hand period, and to the remainder bring down the second period for a dividend.

Double the root for a divisor, and the quotient of the dividend exclusive of its right hand figure, divided by the divisor, is the next figure of the required root; which figure is also to be placed at the right of the divisor.

Multiply the divisor, thus augmented, by the last figure of the root, subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.

Double the root now found for a new divisor and continue the operation as before, until all the periods are brought down.

190. EXAMPLES.

1. Find the square root of 28111204.

Solution. The operation is as follows:

	28111204	5302 = *Ans.*
	25	
1st Rem.	311	103 1st Divisor.
	309	
2d Rem.	212	106 2d Divisor.
3d Rem.	21204	10602 3d Divisor.
	21204	
4th Rem.	0.	

2. Find the square root of 61009. *Ans.* 247

3. Find the square root of 57198969. *Ans.* 7563.

4. Find the square root of 1607448649. *Ans.* 40093.

5. Find the square root of 48303584·206084. *Ans.* 6950·078.

6. Find the square root of 0·000256. *Ans.* 0·016.

7. Find the square root of $\frac{256}{361}$. *Ans.* $\frac{16}{19}$.

8. Find the square root of $1\frac{7}{9}$. *Ans.* $1\frac{1}{3}$.

9. Find the square root of 5. *Ans.* 2·236 +.

10. Find the square root of 101. *Ans.* 10·049 +.

11. Find the square root of 9·6. *Ans.* 3·098 +.

12. Find the square root of 0·003. *Ans.* 0·05477 +.

13. Find the square root of 10. *Ans.* 3·16227 +.

14. Find the square root of 1000. *Ans.* 31·6227 +.

191. *Corollary.* The roots of vulgar fractions and mixed numbers may be computed in decimals by first reducing them to decimals.

192. EXAMPLES.

1. Find the square root of $\frac{1}{17}$ to 4 places of decimals.
Ans. 0·2425 +.

2. Find the square root of $\frac{5}{12}$ to 3 places of decimals.
Ans. 0·645 +.

3. Find the square root of $1\frac{3}{4}$ to 2 places of decimals.
Ans. 1·32 +.

4. Find the square root of $11\frac{11}{16}$ to 3 places of decimals.
Ans. 3·418 +

5. Find the 3d root of $\frac{2}{3}$ to 3 places of decimals.
Ans. 0·873 +.

6. Find the 3d root of $\frac{5}{6}$ to 3 places of decimals.
Ans. 0·941 +.

7. Find the 3d root of $15\frac{2}{3}$ to 3 places of decimals.
Ans. 2·502 +.

CHAPTER V.

POWERS AND ROOTS.

SECTION I.

Powers and Roots of Monomials.

193. *Problem. To find any power of a monomial.*

Solution. The rule of art. 28, applied to this case, in which the factors are all equal, gives for the coefficient of the required power the same power of the given coefficient, and for the exponent of each letter the given exponent added to itself as many times as there are units in the exponent of the required power. Hence

Raise the coefficient of the given monomial to the required power; and multiply each exponent by the exponent of the required power.

194. *Corollary.* An even power of a negative quantity is, by art. 32, positive, and an odd power is negative.

195. EXAMPLES.

1. Find the third power of $2\,a^2\,b^5\,c$. *Ans.* $8\,a^6\,b^{15}\,c^3$.
2. Find the mth power of a^n. *Ans.* $a^{m\,n}$.
3. Find the $-m$th power of a^n. *Ans.* $a^{-m\,n}$.
4. Find the mth power of a^{-n}. *Ans.* $a^{-m\,n}$
5. Find the $-m$th power of a^{-n}. *Ans.* $a^{m\,n}$.

6. Find the 6th power of the 5th power of $a^3 b c^2$.
Ans. $a^{90} b^{30} c^{60}$.

7. Find the qth power of the $-p$th power of the mth power of a^{-n}. Ans. a^{mnpq}.

8. Find the rth power of $a^m b^{-n} c^p d$.
Ans. $a^{mr} b^{-nr} c^{pr} d^r$.

9. Find the -3d power of $a^{-2} b^3 c^{-5} f^6 x^{-1}$.
Ans. $a^6 b^{-9} c^{15} f^{-18} x^3$.

10. Find the 4th power of $-\frac{a^4 b^5}{c^3 d f}$. Ans. $\frac{a^{16} b^{20}}{c^{12} d^4 f^4}$.

11. Find the $-2m$th power of the -1st power of $\frac{a^2 b^3}{c d^5}$.
Ans. $\frac{a^{4m} b^{6m}}{c^{2m} d^{10m}}$.

12. Find the 5th power of $-2a^2$. Ans. $-32 a^{10}$.

13. Find the 4th power of $-3 b^{-3}$. Ans $81 b^{-12}$.

14. Find the 5th power of the 4th power of the 3d power of $-a$. Ans. a^{60}.

15. Find the -5th power of the -3d power of $-a$.
Ans. $-a^{15}$.

16. Find the -4th power of the -3d power of $-\frac{a}{b}$.
Ans. $\frac{a^{12}}{b^{12}}$.

196. *To find any root of a monomial.*

Solution. Reversing the rule of art. 193, we obtain immediately the following rule.

Extract the required root of the coefficient; and divide each exponent by the exponent of the required root.

197. *Corollary.* The odd root of a positive quantity is, by art. 194, positive, and that of a negative quantity is negative. The even root of a positive quantity may be either positive or negative, which is expressed by the double sign $\pm$ preceding it. But, since the even powers of all quantities, whether positive or negative, are positive, the even root of a negative quantity can be neither a positive quantity nor a negative quantity, and it is, as it is called, *an imaginary quantity.*

198. *Corollary.* When the exponent of a letter is not exactly divisible by the exponent of the root to be extracted, a *fractional exponent* is obtained, which may consequently be used to represent the radical sign.

199. EXAMPLES.

1. Find the mth root of a^{mn}. *Ans.* a^n.

2. Find the mth root of a^{-mn} *Ans.* a^{-n}.

3. Find the square root of $9\,a^4\,b^2 f^{-12} g^{-8n}$.
Ans. $\pm 3\,a^2\,b f^{-6} g^{-4n}$.

4. Find the 4th root of $\frac{a^8\,b^{20}\,c^4}{16\,d^{12}z^{16}}$. *Ans.* $\pm\frac{a^2\,b^5\,c}{4\,d^3z^4}$.

5. Find the 9th root of $-2^{36}\,a^{45}\,b^9$. *Ans.* $-2^4\,a^5\,b$.

6. Find the mth root of a^n.
Ans. $a^{\frac{n}{m}}$.

7. Find the mth root of $\frac{1}{a^n}$. *Ans.* $\frac{1}{a^{\frac{n}{m}}} = a^{-\frac{n}{m}}$

8. Find the mth root of $\frac{a^n b^p}{c^r e^t}$. *Ans.* $a^{\frac{n}{m}} b^{\frac{p}{m}} c^{-\frac{r}{m}} e^{-\frac{t}{m}}$.

9. Find the 5th root of $-a^3$. *Ans.* $-a^{\frac{3}{5}}$.

10. Find the square root of a. *Ans.* $a^{\frac{1}{2}}$.

11. Find the 3d root of $-a$. *Ans.* $-a^{\frac{1}{3}}$.

12. Find the mth root of a. *Ans.* $a^{\frac{1}{m}}$.

200. *Corollary.* By taking out -1 as the factor of a negative quantity, of which an even root is to be extracted, the root of each factor may be extracted separately.

201. EXAMPLES.

1. Find the square root of $-a^2$. *Ans.* $a\sqrt{-1}$.

2. Find the 4th root of $-a^4 b^8 c^3$. *Ans.* $a b^2 c^{\frac{3}{4}} \sqrt[4]{-1}$.

3. Find the 8th root of $-a$. *Ans.* $a^{\frac{1}{8}} \sqrt[8]{-1}$

SECTION II.

Calculus of Radical Quantities.

202. *Most of the difficulties in the calculation of radical quantities will be found to disappear if fractional exponents are substituted for the radical signs, and if the rules, before given for exponents, are applied to fractional exponents.*

In the results thus obtained, radical signs may again be substituted for the fractional exponents;

but, before this substitution is made, the fractional exponents in each term should be reduced to a common denominator, in order that one radical sign may be sufficient for each term.

When numbers occur under the radical sign, they should be separated into their factors, and the roots of these factors should be extracted as far as possible.

Fractional exponents greater than unity should often be reduced to mixed numbers.

203. EXAMPLES.

1. Add together $7\sqrt[3]{54\,a^3\,b^5\,c^3}$ and $3\sqrt[3]{16\,a^3\,b^5\,c^3}$

Solution. We have

$$7\sqrt[3]{54\,a^3\,b^5\,c^3} = 7\sqrt[3]{2\,.\,3^3\,.\,a^3\,b^5}\,c = 7\,.\,2^{\frac{1}{3}}\,.\,3\,.\,a\,b^{\frac{5}{3}}\,c$$
$$= 21\,.\,2^{\frac{1}{3}}\,.\,a\,b^{1+\frac{2}{3}}\,c = 21\,.\,2^{\frac{1}{3}}\,a\,b\,b^{\frac{2}{3}}\,c$$
$$= 21\,a\,b\,c\sqrt[3]{2\,b^2}.$$

$$3\sqrt[3]{16\,a^3\,b^5\,c^3} = 3\sqrt[3]{2^4\,a^3\,b^5}\,c = 3\,.\,2^{\frac{4}{3}}\,.\,a\,b^{\frac{5}{3}}\,c$$
$$= 3\,.\,2\,.\,2^{\frac{1}{3}}\,a\,b\,b^{\frac{2}{3}}\,c = 6\,a\,b\,c\sqrt[3]{2\,b^2},$$

whence

$$7\sqrt[3]{54\,a^3\,b^5\,c^3} + 3\sqrt[3]{16\,a^3\,b^5\,c^3} = 21\,a\,b\,c\sqrt[3]{2\,b^2} + 6\,a\,b\,c\sqrt[3]{2\,b^2}$$
$$= 27\,a\,b\,c\sqrt[3]{2\,b^2}.$$

2. From the sum of $\sqrt{24}$ and $\sqrt{54}$ subtract $\sqrt{6}$.

Ans. $4\sqrt{6}$.

3. From the sum of $\sqrt{45\,c^3}$ and $\sqrt{5\,a^2\,c}$ subtract $\sqrt{80\,c^3}$.

Ans. $(a-c)\sqrt{5\,c}$.

Examples in the Calculus of Radical Quantities.

4. Find the continued product of $\sqrt[m]{a}$, $\sqrt[m]{b}$, and $\sqrt[m]{c}$.
Ans. $\sqrt[m]{a\,b\,c}$.

5. Find the continued product of $a\sqrt[n]{x}$, $b\sqrt[n]{y}$, $c\sqrt[n]{z}$.
Ans. $a\,b\,c\sqrt[n]{x\,y\,z}$.

6. Multiply $c\sqrt{a}$ by $b\sqrt{a}$. *Ans.* $a\,b\,c$.

7. Multiply $\sqrt[3]{a}$ by $\sqrt[4]{a}$. *Ans.* $\sqrt[12]{a^7} = a^{\frac{7}{12}}$.

8. Multiply $\sqrt[m]{a}$ by $\sqrt[n]{a}$. *Ans.* $a^{\frac{m+n}{mn}} = \sqrt[mn]{a^{m+n}}$.

9. Multiply $\sqrt[4]{a^3}$ by $\sqrt[3]{a^5}$. *Ans.* $a^{\frac{29}{12}} = a^2\sqrt[12]{a^5}$.

10. Find the continued product of $a^{-\frac{1}{2}}$, $a^{\frac{7}{4}}$, $a^{-\frac{1}{5}}$.
Ans. $a^{\frac{21}{20}} = a\sqrt[20]{a}$.

11. Multiply $b^{-2}\sqrt[4]{a^{-3}}$ by $a^{\frac{5}{6}}\,b^{\frac{1}{2}}\,c$.
Ans. $a^{\frac{1}{12}}\,b^{-\frac{3}{2}}\,c = \frac{c}{b}\sqrt[12]{\frac{a}{b^6}}$.

12. Multiply $\frac{a}{b^{\frac{1}{2}}\,c^{\frac{3}{4}}}$ by $\frac{a^{\frac{7}{8}}\,b}{c^{-\frac{1}{2}}}$.
Ans. $a^{\frac{15}{8}}\,b^{\frac{1}{2}}\,c^{-\frac{1}{4}} = a^2\sqrt[8]{\frac{b^4}{a\,c^2}}$.

13. Multiply $3+\sqrt{5}$ by $2-\sqrt{5}$. *Ans.* $1-\sqrt{5}$.

14. Multiply $7+2\sqrt{6}$ by $9-5\sqrt{6}$.
Ans. $3-17\sqrt{6}$.

15. Multiply $13-\sqrt{5}$ by $7+3\sqrt{5}$.
Ans. $76+32\sqrt{5}$.

16. Multiply $\frac{3}{2}+\frac{5}{2}\sqrt{\frac{1}{2}}$ by $\frac{1}{2}-7\sqrt{\frac{1}{2}}$.
Ans. $-8-\frac{37}{4}\sqrt{\frac{1}{2}}$.

17. Multiply $-5-\sqrt{\frac{2}{3}}$ by $-5+\sqrt{\frac{2}{3}}$. *Ans.* $24\frac{1}{3}$.

Examples in the Calculus of Radical Quantities.

18. Multiply $9+2\sqrt{10}$ by $9-2\sqrt{10}$. *Ans.* 41.

19. Multiply $2\sqrt{8}+3\sqrt{5}-7\sqrt{2}$ by $\sqrt{72}-5\sqrt{20}-2\sqrt{2}$. *Ans.* $-174+42\sqrt{10}$.

20. Multiply $a+\sqrt{b}$ by $a-\sqrt{b}$. *Ans.* a^2-b.

21. Multiply $\sqrt{a}+\sqrt{b}$ by $\sqrt{a}-\sqrt{b}$. *Ans.* $a-b$.

22. Multiply $\sqrt{a}+c\sqrt[3]{b}$ by $\sqrt{a}-c\sqrt[3]{b}$. *Ans.* $a-c^2\sqrt[3]{b^2}$.

23. Multiply $\sqrt[4]{a^3}+\sqrt[5]{b^2}$ by $\sqrt[4]{a^3}-\sqrt[5]{b^2}$. *Ans.* $a\sqrt{a}-\sqrt[5]{b^4}$.

24. Divide $\sqrt[m]{a}$ by $\sqrt[m]{b}$. *Ans.* $\sqrt[m]{\frac{a}{b}}$.

25. Divide a by $\sqrt{a}$. *Ans.* $\sqrt{a}$.

26. Divide $2ab^2c^3$ by $4\sqrt[3]{a^3bc^5d}$. *Ans.* $\frac{1}{2}\sqrt[3]{\frac{b^5c^4}{d}}$.

27. Divide $\sqrt[3]{a^2bc}$ by $\sqrt[5]{ab^2c^3}$. *Ans.* $\sqrt[15]{\frac{a^7}{bc^4}}$.

28. Divide $\sqrt[4]{\frac{a}{b}}$ by $\sqrt{\frac{a}{b}}$. *Ans.* $\sqrt[4]{\frac{b}{a}}$.

29. Divide $a^{\frac{m}{n}}$ by $a^{\frac{p}{q}}$. *Ans.* $a^{\frac{mq-np}{nq}}$.

30. Divide $ca^{\frac{3}{4}}$ by $da^{\frac{5}{6}}$. *Ans.* $\frac{c}{d\sqrt[12]{a}}$.

31. Divide $a^{\frac{3}{5}}b^{\frac{1}{2}}$ by $a^{-\frac{7}{5}}b^{-\frac{1}{4}}c$. *Ans.* $\frac{a^2\sqrt[4]{b^3}}{c}$.

32. Divide $\frac{a^{-\frac{9}{2}}b^{\frac{2}{3}}}{c^{\frac{1}{6}}d^3}$ by $\frac{a^{-\frac{29}{4}}d^{\frac{11}{3}}}{b^{\frac{8}{5}}c}$ *Ans.* $\frac{a^{\frac{11}{4}}b^{\frac{34}{15}}c^{\frac{5}{6}}}{d^{\frac{20}{3}}}$.

To free an Equation from Radical Quantities.

204. ***Problem.*** *To free an equation from radical quantities.*

First Method of Solution. Free the equation from fractions, as in art. 112.

Bring all the terms multiplied by either of the radical quantities, whether they contain other radical quantities or not, to the first member, and all the other terms to the second member of the equation. Raise both members of the equation to that power, which is of the same degree with the root of the radical factor of the first member, and this radical factor will be made to disappear; and by performing the same process on the new equation thus formed, either of the other radical quantities may be made to disappear, and in most cases which occur in practice it will be found that the equation can, in this way, be freed from radical quantities.

205. EXAMPLES.

1. Free the equation

$$(a+x)^{\frac{1}{2}}=(b+x)^{\frac{1}{2}}-(c+x)^{\frac{1}{2}}$$

from radical quantities.

Solution. The square of this equation is

$$a+x=b+x-2(b+x)^{\frac{1}{2}}(c+x)^{\frac{1}{2}}+c+x;$$

hence, by transposition,

$$2(b+x)^{\frac{1}{2}}(c+x)^{\frac{1}{2}}=x-a+b+c,$$

the square of which is

$$4(b+x)(c+x)=(x-a+b+c)^2.$$

2. **Solve the equation**

$$\sqrt[m]{x} = a.$$

Ans. $x = a^m$.

3. **Solve the equation**

$$5\,(9\,x - 1)^{\frac{1}{3}} = 2\,(121\,x + 4)^{\frac{1}{3}}.$$

Ans. $x = 1$.

4. **Solve the equation**

$$(16 + x^2)^{\frac{1}{2}} - x = 2.$$

Ans. $x = 3$.

5. **Solve the equation**

$$(21 + 4\,x)^{\frac{1}{2}} = (3 + x)^{\frac{1}{2}} + (8 + x)^{\frac{1}{2}}.$$

Ans. $x = 1$.

6. **Free the equation**

$$(7 + x^2)^{\frac{1}{4}} = (x - 2)^{\frac{1}{2}} + 1$$

from radical quantities.

Ans. $4\,x^3 - 17\,x^2 + 34\,x = 57$.

206. *Scholium.* There are cases, however, in which the preceding method of solution is inapplicable on account of the new radical quantities which are introduced by raising the second member to the required power; but in all cases the following method will be found successful.

207. *Second Method of solving the problem of art.* 204. *Place each of the radical quantities equal to some letter not before used in the equation, and raise the equations thus formed to that power which is of the same degree with the root of its radical quantity, and substitute in the given equation for each radical quantity the corresponding letter. If, then, each letter, thus introduced, is considered to represent a*

new unknown quantity, the new equations, thus formed, are of the same number with that of their unknown quantities; and, since they are free from radical quantities, all their unknown quantities but one can be eliminated by the method of art. 155.

208. EXAMPLES.

1. Free the equation

$$(x^2+x+1)^{\frac{1}{3}}-(x^2-x+1)^{\frac{1}{3}}=1$$

from radical quantities.

Solution. Place

$$y=(x^2+x+1)^{\frac{1}{3}},$$
$$z=(x^2-x+1)^{\frac{1}{3}};$$

whence

$$y^3=x^2+x+1,$$
$$z^3=x^2-x+1;$$

and the given equation becomes

$$y-z=1.$$

If y and z are eliminated between these three equations, the resulting equation is

$$27\,x^4-8\,x^3+39\,x^2-6\,x+28=0.$$

2. Free the equation

$$(x+x^2)^{\frac{1}{3}}+(1+x^2)^{\frac{1}{2}}=1$$

from radical quantities.

Ans. $x^6+5\,x^4+4\,x^3+7\,x^2+8\,x=0.$

209. When, in an equation, the same quantity is affected by different radical signs, these radical signs, expressed by fractional exponents, may be reduced

to a common denominator, and, if a letter is placed equal to that root of this quantity, which is of a degree represented by the common denominator, these different radical quantities may be represented by the powers of this letter.

210. EXAMPLES.

1. Free the equation

$$(a+x)^{\frac{2}{3}}+A\,(a+x)^{\frac{1}{2}}+B\,(a+x)^{\frac{1}{3}}+C\,(a+x)^{\frac{1}{4}}=0$$

from radical quantities.

Solution. This equation becomes, by reducing all its fractional exponents to the same denominator,

$$(a+x)^{\frac{8}{12}}+A\,(a+x)^{\frac{6}{12}}+B\,(a+x)^{\frac{4}{12}}+C\,(a+x)^{\frac{3}{12}}=0;$$

whence, if we place

$$y=(a+x)^{\frac{1}{12}}, \text{ or } y^{12}=a+x,$$

we have

$$y^8+A\,y^6+B\,y^4+C\,y^3=0,$$

or

$$y^5+A\,y^3+B\,y\;+C=0;$$

the solution of which gives the value of y, which, being substituted in

$$x=y^{12}-a,$$

gives that of x.

2. Free the equation

$$(x+x^3)^{\frac{1}{3}}+(x+x^3)+(x+x^3)^{\frac{1}{2}}=a$$

from radical quantities.

Ans. From the equation $y^6+y^3+y^2=a$ obtain the value of y, and substitute it in $x^3+x=y^6$.

3. Solve the equation

$$(2x+13)^{\frac{2}{3}} = 3\,(2x+13)^{\frac{1}{2}}.$$

Ans. $x = 358.$

4. Solve the equation

$$\left(5 + (3+x)^{\frac{2}{3}}\right)^{\frac{1}{2}} = 1 + (3+x)^{\frac{1}{3}}$$

Ans. $x = 5.$

SECTION III.

Powers of Polynomials.

211. ***Problem.*** *To find any power of a binomial.*

Solution. This power might be obtained directly by multiplication, but the operation is long and tedious, and can be avoided by a process invented by Newton. To obtain this process, let the given binomial be $a+x$, let n be the exponent of the power, and let the product be arranged according to the powers of x, so that

$$(a+x)^n = A + B\,x + C\,x^2 + D\,x^3 + E\,x^4 + \&c.,$$

in which the coefficients, A, B, C, &c., are to be determined; none but positive integral powers of x are written in the second member, because the product could, evidently, give no others, and all the positive integral powers of x are included, because the coefficients of any which are superfluous must be found to vanish.

First. To find the value of A. Let

$$x = 0$$

and the development becomes

$$a^n = A.$$

Secondly. To find the form in which a enters into the development. Let

$$a = 1,\ x = x',$$

and let the corresponding values of a^n, B, C, &c., be 1, B', C', &c., and we have

$$(1+x')^n = 1 + B'x' + C'x'^2 + D'x'^3 + \&c.$$

in which A', B', C', &c., must be independent of a. The product of this equation by a^n is

$$a^n(1+x')^n = (a+ax')^n = a^n + B'a^n x' + C'a^n x'^2 + \&c.,$$

in which, if we put

$$ax' = x, \text{ or } x' = \frac{x}{a},$$

we have

$$a^n x' = a^{n-1}x,\ a^n x'^2 = a^{n-2}x^2,\ \&c.,$$

and

$$(a+x)^n = a^n + B'a^{n-1}x + C'a^{n-2}x^2 + D'a^{n-3}x^3 + \&c.$$

Thirdly. To find the coefficients. The derivative of the last equation is, by examples 4 and 8 of art. 175,

$$n(a+x)^{n-1} = B'a^{n-1} + 2C'a^{n-2}x + 3D'a^{n-3}x^2 + 4E'a^{n-4}x^3 + \&c.,$$

which, multiplied by $(a+x)$, gives

$$\begin{aligned} n(a+x)^n = B'a^n &+ 2C'a^{n-1}x + 3D'a^{n-2}x^2 + 4E'a^{n-3}x^3 + \&c \\ &+ B'a^{n-1}x + 2C'a^{n-2}x^2 + 3D'a^{n-3}x^3 + \&c. \end{aligned}$$

The product of $(a+x)^n$ by n gives, also,

$$n(a+x)^n = na^n + nB'a^{n-1}x + nC'a^{n-2}x^2 + nD'a^{n-3}x^3 + \&c.$$

which, compared with the preceding equation, gives, by art. 163,

$$B' = n,$$

$$\begin{array}{lll} 2C' + B' = nB', \text{ or } 2C' = (n-1)B', & \text{or } C' = \frac{1}{2}(n-1)B'; \\ 3D' + 2C' = nC', \quad 3D' = (n-2)C', & D' = \frac{1}{3}(n-2)C'; \\ 4E' + 3D' = nD', \quad 4E' = (n-3)D', & E' = \frac{1}{4}(n-3)D'; \\ \&c. \qquad \&c. & \&c. \end{array}$$

The combination of these results gives

Sir Isaac Newton's Binomial Theorem.

The first term of any power of a binomial is the same power of the first term of the binomial.

In the following terms of the power the exponent of the first term continually decreases by unity whereas the exponent of the second term of the binomial, which is unity in the second term of the power continually increases by unity.

The coefficient of the second term of the power is the exponent of the power.

If the coefficient of any term is multiplied by the exponent which the first term of the binomial has in that term, and divided by the place of the term, the result is the coefficient of the next following term.

212. *Corollary.* The equations of the preceding article give

$$B' = n$$

$$C' = \frac{n\,(n-1)}{1\,.\,2}$$

$$D' = \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}$$

&c.

Hence

$$(a+x)^n = a^n + n\,a^{n-1}\,x + \frac{n\,(n-1)}{2}\,a^{n-2}x^2 +$$

$$\frac{n(n-1)(n-2)}{2\quad 3}\,a^{n-3}x^3 + \frac{n(n-1)(n-2)(n-3)}{2\quad 3\quad 4}\,a^{n-4}x^4 + \&c.$$

213. *Corollary.* If x is changed into $-x$ in the preceding formula, it becomes

$$(a-x)^n = a^n - n\,a^{n-1}\,x + \frac{n\,(n-1)}{2}\,a^{n-2}\,x^2 -$$
$$\frac{n\,(n-1)\,(n-2)}{2\ .\ 3}\,a^{n-3}\,x^3 + \&c.$$

the signs of every other term being reversed.

214. *Corollary.* The preceding formula, written in the reverse order of its terms, gives

$$(x+a)^n = x^n + n\,a\,x^{n-1} + \frac{n\,(n-1)}{2}\,a^2\,x^{n-2} + \&c.$$

whence it appears that

The coefficients of two terms which are equally distant, the one from the first term, and the other from the last term, are equal.

215. EXAMPLES.

1. Find the 6th power of $\frac{2\,a\,c}{b^2} - \frac{1}{4}\,b\,c^2\,d$.

Solution. Place $x = \frac{2\,a\,c}{b^2} = 2\,a\,b^{-2}\,c,$

$$y = \tfrac{1}{4}\,b\,c^2\,d;$$

and we have

$$\left(\frac{2\,a\,c}{b^2} - \tfrac{1}{4}\,b\,c^2\,d\right)^6 = (x-y)^6.$$

But, by the above formula,

$$(x-y)^6 = x^6 - 6\,x^5\,y + 15\,x^4\,y^2 - 20\,x^3\,y^3 + 15\,x^2\,y^4$$
$$-6\,x\,y^5 + y^6;$$

in which, if we substitute the values of x and y, we have

$(2\,a\,b^{-2}\,c - \frac{1}{4}\,b\,c^{2}\,d)^6 = 64\,a^6\,b^{-12}\,c^6 - 48\,a^5\,b^{-9}\,c^7\,d +$
$15\,a^4\,b^{-6}\,c^8\,d^2 - \frac{5}{2}\,a^3\,b^{-3}\,c^9\,d^3 + \frac{15}{64}\,a^2\,c^{10}\,d^4$
$- \frac{3}{256}\,a\,b^3\,c^{11}\,d^5 + \frac{1}{4096}\,b^6\,c^{12}\,d^6.$

2. Find the 10th power of $a + b$.

Ans. $a^{10} + 10\,a^9\,b + 45\,a^8\,b^2 + 120\,a^7\,b^3 + 210\,a^6\,b^4 + 252\,a^5\,b^5 + 210\,a^4\,b^6 + 120\,a^3\,b^7 + 45\,a^2\,b^8 + 10\,a\,b^9 + b^{10}.$

3. Find the 11th power of $1 - x$.

Ans. $1 - 11\,x + 55\,x^2 - 165\,x^3 + 330\,x^4 - 462\,x^5 + 462\,x^6 - 330\,x^7 + 165\,x^8 - 55\,x^9 + 11\,x^{10} - x^{11}.$

4. Find the 4th power of $5 - 4\,x$.

Ans. $625 - 2000\,x + 2400\,x^2 - 1280\,x^3 + 256\,x^4.$

5. Find the 7th power of $\frac{1}{2}\,x + 2\,y$.

Ans. $\frac{1}{128}\,x^7 + \frac{7}{32}\,x^6\,y + \frac{21}{8}\,x^5\,y^2 + \frac{35}{2}\,x^4\,y^3 + 70\,x^3\,y^4 + 168\,x^2\,y^5 + 224\,x\,y^6 + 128\,y^7.$

6. Find the 4th power of $5\,a^2\,c^2\,d - 4\,a\,b\,d^2$.

Ans. $625\,a^8\,c^8\,d^4 - 2000\,a^7\,b\,c^6\,d^5 + 2400\,a^6\,b^2\,c^4\,d^6 - 1280\,a^5\,b^3\,c^2\,d^7 + 256\,a^4\,b^4\,d^8.$

216. ***Problem.*** *To find any power of a polynomial.*

Solution. Suppose the terms of the given polynomial to be arranged according to the powers of any letter, as x, as follows;

$$a + b\,x + c\,x^2 + d\,x^3 + e\,x^4 + \&c.,$$

in which the successive coefficients are denoted by the successive letters of the alphabet. The following is

Arbogast's rule for finding any power of the polynomial.

The first term of the power is the same power of the first term* a *of the given polynomial.

***The coefficient of* x *in the second term is* b *times the derivative of the first term taken with reference to* a.**

To obtain any other coefficient from the preceding coefficient; let* r *be the letter farthest advanced in the alphabet which is contained in any term of the given coefficient.

Then* r *times the derivative of this term with reference to* q, *is a term of the new coefficient; but this process is obviously inapplicable, or at least useless, when* q *is the last letter of the given polynomial so that* r *is zero.

If the given term contains the preceding letter* p *as well as* q, q *times its derivative with reference to* p, *divided by the increased exponent of* q, *is also a term of the new coefficient.

Thus the term $T\, p^{\eta}\, q^{\theta}$ gives, in the following coefficient, the two terms

$$\theta\, T\, p^{\eta}\, q^{\theta+1}\, r \text{ and } \tfrac{\eta}{\theta+1}\, T\, p^{\eta-1}\, q^{\theta+1}.$$

Proof. First. **The value of the first term is obtained precisely as in the binomial theorem by putting**

$$x = 0.$$

Secondly. **Let V denote the given polynomial, so tha**

$$V = a + b\,x + c\,x^2 \ldots + p\,x^m + q\,x^{m+1} + \&c$$

The derivative of V with reference to p is, then,

$$x^m,$$

and that of V^n is, by art. 172,

$$n\, V^{n-1}\, x^m,$$

the derivative of V with reference to q is

$$x^{\varphi+1},$$

and that of V^n is

$$n\,V^{n-1}\,x^{\varphi+1}.$$

Let, now, the required power be the nth and let

$$V^n = A + B\,x + C\,x^2 \ldots + P\,x^{\varphi} + Q\,x^{\varphi+1} + \&c.$$

and let the derivatives of $A, B, C, \ldots P, Q$, &c., with reference to p be $A', B', C', \ldots P', Q'$, &c., and with reference to q, $A'', B'', C'', \ldots P'', Q''$, &c.; the derivatives of the preceding equation with reference to p and q are

$$n\,V^{n-1}\,x^{\varphi} = A' + B'\,x + C'\,x^2 \ldots + P'\,x^{\varphi} + Q'\,x^{\varphi+1} + \&c.$$
$$n\,V^{n-1} x^{\varphi+1} = A'' + B''\,x + C''\,x^2 \ldots + P''\,x^{\varphi} + Q''\,x^{\varphi+1} + \&c.$$

the first of which, multiplied by x, is

$$n\,V^{n-1}\,x^{\varphi+1} = A'\,x + B'\,x^2 + C'\,x^3 \ldots + P'\,x^{\varphi+1} + Q'\,x^{\varphi+2} + \&c.,$$

which, compared with the other, gives, by art. 163, for the coefficient of $x^{\varphi+1}$,

$$P' = Q'',$$

that is, *the derivative of any coefficient with reference to any letter* p, *is the same with that of the succeeding coefficient with reference to the succeeding letter* q.

Thirdly. Every term of Q, in which q is the letter farthest advanced in the alphabet, such as $T\,p^{\pi}\,q^{\theta}$, must give in Q'' a similar term $\theta\;T\,p^{\pi}\;q^{\theta-1}$, or else, if

$$\theta = 1,$$

a term $T\,p^{\pi}$, in which there is no letter so far advanced as q. Every such term, as it belongs also to P', must be the derivative of a similar term in P, that is, of a term in which p is either the farthest advanced letter, or the next to the farthest advanced letter. The terms of

$$P' = Q''$$

are, then, obtained from those of P by derivation, while those of Q are obtained from those of its derivative Q'' by a process, which is the reverse of derivation, and which, by art. 172, consists in *multiplying by* q, *that is, in increasing its exponent by unity, and dividing by its exponent thus increased.* This process is identical with the last paragraph of the rule in this article, and the three preceding paragraphs refer merely to those cases in which the increased exponent of q is unity, so that the division by it is superfluous.

217. *Corollary.* If x is put equal to unity in the value of

$$(a + b\,x + c\,x^2 + \&c.)^n,$$

we have the value of

$$(a + b + c + \&c.)^n,$$

so that any power of a polynomial, the terms of which contain no common letter, is readily found by multiplying the successive terms, after the first, respectively by x, x^2, x^3, x^4, &c., obtaining the power of the polynomial thus formed, and putting

$$x = 1$$

in the result.

218. EXAMPLES.

1. Find the 5th power of $1 + 2\,x + 3\,x^2 + 4\,x^3$.

Solution. Represent the successive coefficients 1, 2, 3, 4 by a, b, c, and d, so that

$$a = 1,\ b = 2,\ c = 3,\ d = 4;$$

and the given polynomial becomes

$$a + b\,x + c\,x^2 + d\,x^3,$$

Polynomial Theorem.

the fifth power of which, is found by the rule to be

$$a^5 + 5a^4 b x + \left.\begin{array}{l} 5a^4 c \\ +10a^3 b^2 \end{array}\right| x^2 + \left.\begin{array}{l} 5a^4 d \\ +20a^3 b c \\ +10a^2 b^3 \end{array}\right| x^3 + \left.\begin{array}{l} 20a^3 b d \\ +10a^3 c^2 \\ +30a^2 b^2 c \\ +5a b^4 \end{array}\right| x^4$$

$$+\left.\begin{array}{l} 20a^3 c d \\ +30a^2 b^2 d \\ +30a^2 b c^2 \\ +20a b^3 c \\ +b^5 \end{array}\right| x^5 + \left.\begin{array}{l} 10a^3 d^2 \\ +60a^2 b c d \\ +10a^2 c^3 \\ +20a b^3 d \\ +30a b^2 c^2 \\ +5b^4 c \end{array}\right| x^6 + \left.\begin{array}{l} 30a^2 b d^2 \\ +30a^2 c^2 d \\ +60a b^2 c d \\ +20a b c^3 \\ +5b^4 d \\ +10b^3 c^2 \end{array}\right| x^7$$

$$+\left.\begin{array}{l} 30a^2 c d^2 \\ +30a b^2 d^2 \\ +60a b c^2 d \\ +5a c^4 \\ +20b^3 c d \\ +10b^2 c^3 \end{array}\right| x^8 + \left.\begin{array}{l} 10a^2 d^3 \\ +60a b c d^2 \\ +20a c^3 d \\ +10b^3 d^2 \\ +30b^2 c^2 d \\ +5b c^4 \end{array}\right| x^9 + \left.\begin{array}{l} 20a b d^3 \\ +30a c^2 d^2 \\ +30b^2 c d^2 \\ +20b c^3 d \\ +c^5 \end{array}\right| x^{10}$$

$$+\left.\begin{array}{l} 20a c d^3 \\ +10b^2 d^3 \\ +30b c^2 d^2 \\ +5c^4 d \end{array}\right| x^{11} + \left.\begin{array}{l} 5a d^4 \\ +20b c d^3 \\ +10c^3 d^2 \end{array}\right| x^{12} + \left.\begin{array}{l} 5b d^4 \\ +10c^2 d^3 \end{array}\right| x^{13} + 5c d^4 x^{14} + d^5 x^{15}.$$

Now, if we substitute for a, b, c, d, their values, the preceding expression becomes

$$(1 + 2x + 3x^2 + 4x^3)^5 = 1 + 10x + 55x^2 + 220x^3 + 690x^4 + 1772x^5 + 3830x^6 + 7040x^7 + 11085x^8 + 14970x^9 + 17203x^{10} + 16660x^{11} + 13280x^{12} + 8320x^{13} + 3840x^{14} + 1024x^{15}.$$

2. Find the 3d power of $a + bx + cx^2$.

$$\textit{Ans.}\ a^3 + 3a^2 b x + \left.\begin{array}{l} 3a^2 c \\ +3a b^2 \end{array}\right| x^2 + \left.\begin{array}{l} 6a b c \\ +b^3 \end{array}\right| x^3 + \left.\begin{array}{l} 3a c^2 \\ +3b^2 c \end{array}\right| x^4 + 3b c^2 x^5 + c^3 x^6.$$

3. Find the 6th power of $a + b + c$.

Ans. $a^6 + 6a^5 b + 6a^5 c + 15a^4 b^2 + 30a^4 b c + 20a^3 b^3 + 15a^4 c^2 + 60a^3 b^2 c + 15a^2 b^4 + 60a^3 b c^2 +$

$60\,a^2b^3c + 6\,ab^5 + 20\,a^3c^3 + 90\,a^2b^2c^2 + 30\,ab^4c + b^6 + 60\,a^2bc^3 + 60\,ab^3c^2 + 6\,b^5c + 15\,a^2c^4 + 60\,ab^2c^3 + 15\,b^4c^2 + 30\,abc^4 + 20\,b^3c^3 + 6\,ac^5 + 15\,b^2c^4 + 6\,bc^5 + c^6.$

4. Find the 4th power of $a^3 - a^2x + ax^2 - x^3$.

Ans. $a^{12} - 4\,a^{11}x + 10\,a^{10}x^2 - 20\,a^9x^3 + 31\,a^8x^4 - 40\,a^7x^5 + 44\,a^6x^6 - 40\,a^5x^7 + 31\,a^4x^8 - 20\,a^3x^9 + 10\,a^2x^{10} - 4\,ax^{11} + x^{12}.$

5. Find the square of $a + bx + cx^2 + dx^3 + ex^4 + fx^5$.

Ans. $$a^2 + 2abx + \left.\begin{array}{r}2ac\\ +\ b^2\end{array}\right| x^2 + \left.\begin{array}{r}2ad\\ +2bc\end{array}\right| x^3 + \left.\begin{array}{r}2ae\\ +2bd\\ +\ c^2\end{array}\right| x^4 + \left.\begin{array}{r}2af\\ +2be\\ +2cd\end{array}\right| x^5 \left.\begin{array}{r}+2bf\\ +2ce\\ +\ d^2\end{array}\right| x^6 + \left.\begin{array}{r}2cf\\ +2de\end{array}\right| x^7 + \left.\begin{array}{r}2df\\ +\ e^2\end{array}\right| x^8 + 2efx^9 + f^2x^{10}.$$

SECTION IV

Roots of Polynomials.

219. ***Problem.*** *To find any root of a polynomial.*

Solution. If the root is arranged according to the powers of either of its letters as x, whether ascending or descending, it is evident from the rule of art. 216, that

The term of the required root which contains the highest power of x, *is found by extracting the root of the corresponding term of the given polynomial.*

If, now, R is the first portion of the root, and R' the rest of it, and if P is the given polynomial of which the nth root is to be extracted, we have

$$P = (R + R')^n = R^n + n\,R^{n-1}R' + \&c.$$

or

$$P - R^n = n\,R^{n-1}\,R' + \&c.$$

and

$$\frac{P - R^n}{n\,R^{n-1}} = R' + \&c.$$

and if, in $P - R^n$ and $n\,R^{n-1}$, only the first term is retained, the first term of the quotient is the first term of R'; and a new portion of the root is thus found, which, combined with those before found, gives a new value of R and of $P - R^n$, which, divided by the value of $n\,R^{n-1}$ already obtained, gives a new term of the root, and so on.

Hence to obtain the second term of the root, raise the first term of the root to the power denoted by the exponent of the root, and subtract the result from the given polynomial, bringing down only the first term of the remainder for a dividend.

Also raise the first term of the root to the power denoted by the exponent one less than that of the root, and multiply this power by the exponent of the root for a divisor.

Divide the dividend by the divisor, and the quotient is the second term of the root.

The next term is found, by raising the root already found to the power denoted by the exponent of the required root, subtracting this power from the given polynomial, and dividing the first term of the remainder by the divisor used for obtaining the second term.

This divisor, indeed, being once obtained, is to be used in each successive division, the successive dividends being the first terms of the successive remainders.

220. EXAMPLES.

1. Find the 4th root of $81\,x^8 - 216\,x^7 + 336\,x^5 - 56\,x^4 - 224\,x^3 + 64\,x + 16$.

Solution. The operation is as follows, in which the root is written at the left of the given power, and the divisor at the left of each dividend or remainder; and only the first term of each remainder is brought down.

$$81x^8-216x^7+336x^5-56x^4-224x^3+64x+16 \mid 3x^2-2x-2.$$
$$81\,x^8$$

1st Rem. $-216\,x^7 \mid 108\,x^6 = 4 \times (3\,x^2)^3$

$$81\,x^8-216\,x^7+216\,x^6-96\,x^5+16\,x^4=(3\,x^2-2\,x)^4$$

2d Rem. $\quad -216\,x^6 \quad \mid 108\,x^6$

$$81x^8-216x^7+336x^5-56x^4-224x^3+64x+16=(3x^2-2x-2)^4$$

3d Rem. $\quad 0.$

2. Find the 3d root of $a^3 + 3\,a^2\,b + 3\,a^2\,c + 3\,a\,b^2 + 6\,a\,b\,c + 3\,a\,c^2 + b^3 + 3\,b^2\,c + 3\,b\,c^2 + c^3$.

Ans. $a + b + c$.

3. Find the 3d root of $a^3 + 6\,a^2\,b - 3\,a^2\,c + 12\,a\,b^2 - 12\,a\,b\,c + 3\,a\,c^2 + 8\,b^3 - 12\,b^2\,c + 6\,b\,c^2 - c^3$.

Ans. $a + 2\,b - c$.

4. Find the 3d root of $343\,x^6 - 441\,x^5\,y + 777\,x^4\,y^2 - 531\,x^3\,y^3 + 444\,x^2\,y^4 - 144\,x\,y^5 + 64\,y^6$.

Ans. $7\,x^2 - 3\,x\,y + 4\,y^2$.

5. Find the 4th root of $81\,a^4 - 540\,a^3\,b - 72\,a^3\,c + 1350\,a^2\,b^2 + 360\,a^2\,b\,c + 24\,a^2\,c^2 - 1500\,a\,b^3 - 600\,a\,b^2\,c - 80\,a\,b\,c^2 - \frac{32}{9}\,a\,c^3 + 625\,b^4 + \frac{1000}{3}\,b^3\,c + \frac{200}{3}\,b^2\,c^2 + \frac{160}{27}\,b\,c^3 + \frac{16}{81}\,c^4$.

Ans. $3\,a - 5\,b - \frac{2}{3}\,c$.

6. Find the 5th root of $16807\,a^{10}\,b^5 - \frac{12005}{2}\,a^8\,b^4 + \frac{1715}{2}\,a^6\,b^3 - \frac{24010}{3}\,a^4\,b^7\,c - \frac{245}{4}\,a^4\,b^2 - \frac{6860}{3}\,a^2\,b^6\,c + \frac{35}{16}\,a^2\,b + 245\,b^5\,c - \frac{1}{32} + \frac{13720}{9}\,a^{-2}\,b^9\,c^2 - \frac{35}{3}\,a^{-2}\,b^4\,c - \frac{980}{3}\,a^{-4}\,b^8\,c^2 + \frac{5}{24}\,a^{-4}\,b^3\,c + \frac{70}{3}\,a^{-6}\,b^7\,c^2 + \frac{3920}{27}\,a^{-6}\,b^{11}\,c^3 - \frac{5}{9}\,a^{-8}\,b^6\,c^2 - \frac{560}{27}\,a^{-10}\,b^{10}\,c^3 + \frac{20}{27}\,a^{-12}\,b^5\,c^3 + \frac{560}{81}\,a^{-14}\,b^{13}\,c^4 - \frac{40}{81}\,a^{-16}\,b^{12}\,c^4 + \frac{32}{243}\,a^{-20}\,b^{15}\,c^5$.

Ans. $7\,a^2\,b - \frac{1}{2} + \frac{2}{3}\,a^{-4}\,b^3\,c$.

7. Find the 9th root of $y^{27} + 27\,y^{25} + 324\,y^{23} + 2268\,y^{21} + 10206\,y^{19} + 30618\,y^{17} + 61236\,y^{15} + 78732\,y^{13} + 59049\,y^{11} + 19683\,y^9$.

Ans. $y^3 + 3\,y$.

221. *Corollary.* When the preceding method is applied to the extraction of the square root, it admits of modifications similar to those of art. 189, and we have the following rule

To extract the square root of a given polynomial.

Arrange its terms according to the powers of some letter, extract the square root of the first term for the first term of the root.

Double the part of the root thus found for a divisor, subtract the square of this part of the root from the given polynomial, and divide the first term of the remainder by the divisor; the quotient is the second term of the root.

Double the terms of the root already found for a new divisor; subtract from the preceding remainder the product of the last term of the root multiplied by the preceding divisor augmented by the last term of the root. Divide the first term of this new remainder by the first term of the corresponding divisor, and the quotient is the next term of the root.

Proceed in the same way, to find the other terms of the root.

222. EXAMPLES.

1. Find the square root of $x^6+4x^5+20x^2-16x+16$.

Solution. In the following solution, the arrangement is similar to that in the example of art. 190.

$$
\begin{array}{r|l}
x^6+4x^5+20x^2-16x+16 & x^3+2x^2-2x+4. \textit{ Ans.} \\
x^6 \qquad\qquad\qquad\qquad\qquad & \\
\hline
4x^5+20x^2-16x+16 & 2x^3 \\
4x^5+\ \ 4x^4 \qquad\qquad\qquad & \\
\hline
-4x^4+20x^2-16x+16 & 2x^3+4x^2 \\
-4x^4-\ \ 8x^3+\ \ 4x^2 \qquad\qquad & \\
\hline
8x^3+16x^2-16x+16 & 2x^3+4x^2-4x \\
8x^3+16x^2-16x+16 & \\
\hline
0. \qquad\qquad &
\end{array}
$$

2. Find the square root of $25a^4-30a^3b+49a^2b^2-24ab^3+16b^4$. *Ans.* $5a^2-3ab+4b^2$.

3. Find the square root of $4x^6+12x^5+5x^4-2x^3+7x^2-2x+1$. *Ans.* $2x^3+3x^2-x+1$.

4. Find the square root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$. *Ans.* a^2-ax+x^2.

5. Find the square root of $\frac{9}{4}+6x-17x^2-28x^3+49x^4$. *Ans.* $\frac{3}{2}+2x-7x^2$.

SECTION V.

Binomial Equations.

223. *Definition.* When an equation with one unknown quantity is reduced to a series of monomials, and all its terms which contain the unknown quantity are multiplied by the same power of the unknown quantity, *it may be represented by the general form*

$$A x^n + M = 0,$$

and may be called a binomial equation.

224. ***Problem.*** ***To solve a binomial equation.***

Solution. Suppose the given equation to be

$$A x^n + M = 0.$$

Transposing M and dividing by A, we have

$$x^n = -\frac{M}{A},$$

the nth root of which is

$$x = \sqrt[n]{-\frac{M}{A}}.$$

Hence, *find the value of the power of the unknown quantity which is contained in the given equation, precisely as if this power were itself the unknown quantity, and the given equations were of the first degree. Extract that root of the result which is denoted by the index of the power.*

225. *Corollary.* Equations containing two or more unknown quantities will often, by elimination, conduct to binomial equations.

226. EXAMPLES.

1. Solve the two equations

$$x y^7 + 2 y^7 - 4 y^3 - 8 x + 16 = 0,$$
$$x^2 y^7 - 4 y^7 - 4 x y^3 + 8 y^3 + 32 x - 64 = 0.$$

Solution. The elimination of y between these two equations, by the process of art. 155, gives

$$8 x^2 - 32 = 0,$$

whence we have

$$x^2 = 4,$$
$$x = \pm 2.$$

Now the value of x,

$$x = + 2,$$

being substituted in the first of the given equations, produces

$$4 y^7 - 4 y^3 = 0,$$

which is satisfied by the value of y,

$$y = 0;$$

or if we divide by $4 y^3$, we have

$$y^4 - 1 = 0,$$
$$y^4 = 1,$$
$$y = \sqrt[4]{1} = \pm 1 \text{ or } = \pm \sqrt{-1},$$

as will be shown when we treat of the theory of equations

Again, the value of x,

$$x = - 2,$$

being substituted in the first of the given equations, produces

$$- 4 y^3 + 32 = 0,$$

whence we have

$$y^3 = 8,$$
$$y = 2 \text{ or } = - 1 \pm \sqrt{-3},$$

as will be shown in the theory of equations.

Examples of Binomial Equations.

2. Solve the equation

$$3x^2 + 2x = x^2 + 2x + 18.$$

Ans. $x = \pm 3.$

3. Solve the equation

$$\frac{2x-7}{x-1} = \frac{x+1}{2x+7}.$$

Ans. $x = \pm 4.$

4. Solve the equation

$$x + \frac{1}{x} = \frac{26}{x^2 - x} - 1.$$

Ans. $x = 3.$

5. Solve the equation

$$\frac{x^3 + x + 8}{x^3 + 4} + \frac{x^3 + x - 8}{x^3 - 4} = 2.$$

Ans. $x = \pm 2.$

6. Solve the equation

$$\sqrt{(2x + 2)} = x + 1.$$

Ans. $x = \pm 1.$

7. Solve the equation

$$\sqrt[3]{(x^5 - 81x + 1)} = 1.$$

Ans. $x = 0$, or $x = \pm 3.$

8. Solve the two equations

$$x^3 + y^5 = 2a,$$
$$x^3 - y^5 = 2b,$$

Ans. $x = \sqrt[3]{(a + b)}, y = \sqrt[5]{(a - b)}.$

9. Solve the two equations

$$y^6 - 33y^3 + x^4 - 17x^2 = 0,$$
$$y^6 + 17y^3 + x^4 - 33x^2 = 0.$$

Ans. $x = 0$, and $y = 0$; or $x = \pm 5$, and $y = 2.$

Examples of Binomial Equations.

10. What number is it, whose half multiplied by its third part, gives 864? *Ans.* 72.

11. What number is it, whose 7th and 8th parts multiplied together, and the product divided by 3, gives the quotient $298\frac{2}{3}$? *Ans.* 224.

12. Find a number such, that if we first add to it 94, then subtract it from 94, and multiply the sum thus obtained by the difference, the product is 8512. *Ans* 18.

13. Find a number such, that if we first add it to a, then subtract it from a, and multiply the sum by the difference, the product is b. *Ans.* $\sqrt{(a^2-b)}$.

14. Find a number such, that if we first add it to a, then subtract a from it, and multiply the sum by the difference, the product is b. *Ans.* $\sqrt{(a^2+b)}$.

15. What two numbers are they whose product is 750, and quotient $3\frac{1}{3}$? *Ans.* 50 and 15.

16. What two numbers are they whose product is a, and quotient b?

Ans. $\sqrt{a\,b}$ and $\sqrt{\frac{a}{b}}$.

17. What two numbers are they, the sum of whose squares is 13001, and the difference of whose squares is 1449? *Ans.* 85 and 76.

18. What two numbers are they, the sum of whose squares is a, and the difference of whose squares is b?

Ans. $\sqrt{\frac{1}{2}(a+b)}$ and $\sqrt{\frac{1}{2}(a-b)}$.

19. What two numbers are to one another as 3 to 4, the sum of whose squares is 324900? *Ans.* 342 and 456.

20. What two numbers are as m to n, the sum of whose squares is a?

Ans. $\frac{m\sqrt{a}}{\sqrt{(m^2+n^2)}}$ and $\frac{n\sqrt{a}}{\sqrt{(m^2+n^2)}}$.

Examples of Binomial Equations.

21. What two numbers are as m to n, the difference of whose squares is a?
Ans. $\frac{m\sqrt{a}}{\sqrt{(m^2-n^2)}}$ and $\frac{n\sqrt{a}}{\sqrt{(m^2-n^2)}}$.

22. A certain capital is let at 4 per cent.; if we multiply the number of dollars in the capital, by the number of dollars in the interest for 5 months, we obtain $117041\frac{2}{3}$. What is the capital? *Ans.* \$2650.

23. A person has three kinds of goods, which together cost \$5525. The pound of each article costs as many dollars as there are pounds of that article; but he has one third more of the second kind than he has of the first, and $3\frac{1}{2}$ times as much of the third as he has of the second. How many pounds has he of each?
Ans. 15 pounds of the first, 20 of the second, and 70 of the third.

24. Find three numbers such, that the product of the first and second is 6, that of the first and third is 10, and the sum of the squares of the second and third is 34.
Ans. 2, 3, 5.

25. Find three numbers such, that the product of the first and second is a, that of the first and third is b, and that of the second and third is c.
Ans. $\sqrt{\frac{ab}{c}}$, $\sqrt{\frac{ac}{b}}$, and $\sqrt{\frac{bc}{a}}$.

26. What number is it, whose third part, multiplied by its square, gives 1944? *Ans.* 18.

27. What number is it, whose half, third, and fourth, multiplied together, and the product increased by 32, gives 4640? *Ans.* 48.

28. What number is that, whose fourth power divided by $\frac{1}{8}$th of it, and 167 subtracted from the quotient, gives the remainder 12000? *Ans.* $11\frac{1}{2}$.

29. Some merchants engage in business; each contributes a thousand times as many dollars as there are partners. They gain in this business $2560; and it is found that this gain is exactly half their own number per cent. How many merchants are there? *Ans.* 8.

30. Find three numbers such, that the square of the first multiplied by the second is 112; the square of the second multiplied by the third is 588; and the square of the third multiplied by the first is 576. *Ans.* 4, 7, 12.

227. *Corollary.* When the solution of a problem gives for either of its unknown quantities only imaginary values, the problem must be impossible.

228. EXAMPLE.

In what case would the value of the unknown quantity in example 13 of art. 226 be imaginary? and why should the problem in this case be impossible?

Ans. When $b > a^2$,

that is, when the product of the sum and difference is required to be greater than the square of a. Now if the required number is x, this product is

$$(a + x)(a - x) = a^2 - x^2;$$

and, therefore, less than a^2

CHAPTER VI.

EQUATIONS OF THE SECOND DEGREE

229. It may easily be shown, as in art. 120, that *any equation of the second degree with one unknown quantity, may be reduced to the form*

$$A x^2 + B x + M = 0,$$

in which $A x^2$ denotes the aggregate of all the terms multiplied by the second power of the unknown quantity, $B x$ denotes all the terms multiplied by the unknown quantity itself, and M denotes all the terms which do not contain the unknown quantity.

230. *Problem. To solve an equation of the second degree with one unknown quantity.*

Solution. Having reduced the given equation to the form

$$A x^2 + B x + M = 0,$$

we could easily reduce it to an equation of the first degree, by extracting its square root, if the first member were a perfect square.

But this cannot be the case, unless the first term is a perfect square; the equation can, however, always be brought to a form in which its first term is a perfect square, by multiplying it by some quantity which will render the coefficient of the first term a perfect square, multiplying by this coefficient itself, for instance; thus the given equation multiplied by A becomes

$$A^2 x^2 + A B x + A M = 0.$$

Now that the equation is in this form, we can readily ascertain whether its first member is a perfect square, by attempting to extract its root, as follows:

$$\begin{array}{l|l} A^2 x^2 + A B x + A M & A x + \frac{1}{2} B. \quad \text{Root.} \\ A^2 x^2 & \\ \hline \qquad A B x + A M & 2 A x \\ \qquad A B x + \frac{1}{4} B^2 & \\ \hline \qquad A M - \frac{1}{4} B^2 \quad \text{Rem.} & \end{array}$$

so that the first member is a perfect square only when the remainder is zero, that is,

$$A M - \tfrac{1}{4} B^2 = 0;$$

and, in every other case,

$$A x + \tfrac{1}{2} B$$

is the root of the square which differs from it by this remainder, that is,

$$A^2 x^2 + A B x + A N = (A x + \tfrac{1}{2} B)^2 + A M - \tfrac{1}{4} B^2 = 0;$$

or, transposing $A M - \frac{1}{4} B^2$, we have

$$(A x + \tfrac{1}{2} B)^2 = \tfrac{1}{4} B^2 - A M.$$

Now the square root of this last equation is

$$A x + \tfrac{1}{2} B = \pm \sqrt{(\tfrac{1}{4} B^2 - A M)},$$

which, solved as an equation of the first degree, gives

$$x = \frac{-\frac{1}{2} B \pm \sqrt{(\frac{1}{4} B^2 - A M)}}{A}$$

$$= \frac{-B \pm \sqrt{(B^2 - 4 A M)}}{2 A},$$

in which either of the two signs $+$ or $-$, may be used of the double sign $\pm$, and we thus have the two roots of the given equation

$$x = \frac{-B + \sqrt{(B^2 - 4 A M)}}{2 A},$$

and

$$x = \frac{-B - \sqrt{(B^2 - 4\,A\,M)}}{2\,A}.$$

The equation

$$(A\,x + \tfrac{1}{2}\,B)^2 = \tfrac{1}{4}\,B^2 - A\,M,$$

which is the same as

$$A^2\,x^2 + A\,B\,x + \tfrac{1}{4}\,B^2 = \tfrac{1}{4}\,B^2 - A\,M,$$

is obtained immediately from the equation

$$A^2\,x^2 + A\,B\,x + A\,M = 0,$$

by transposing $A\,M$ to the second member, and adding $\frac{1}{4}\,B^2$ to both members. Hence

To solve an equation of the second degree with one unknown quantity.

Reduce it as in arts. 112 *and* 118, *transposing all the terms which contain the unknown quantity to the first member, and the other terms to the second member.*

Multiply the equation by any quantity, (the least is to be preferred,) which will render the coefficient of the second power of the unknown quantity an exact square.

Add to this equation the square of the quotient, arising from the division of the coefficient of the first power of its unknown quantity, by twice the square root of the coefficient of the second power of its unknown quantity.

Extract the square root of the equation thus augmented, and the result is an equation of the first degree, to be solved as in art. 121.

231. *Corollary.* When we have

$$B^2 - 4\,A \cdot M$$

a negative quantity, that is,

$$B^2 < 4\,A\,M,$$

the roots of the given equation are imaginary.

232. *Scholium.* The preceding method of solving quadratic equations gives the *form* of the roots in all cases, but otherwise it has no advantage in the solution of a numerical equation over the solution given in Chapter IV.

The method of art. 182, applied to this case, gives

$$h = -M$$
$$u = A\,x^2 + B\,x,\ U = 2\,A\,x + B,$$

and when

$$x = a$$
$$m = A\,a^2 + B\,a,\ M = 2\,A\,a + B.$$

But the process may be abbreviated precisely as in the case of the square root in art. 189, by observing that

$$\begin{aligned} A(a+h)^2 + B(a+h) &= A\,a^2 + B\,a + (2\,A\,a + B + A\,h)h \\ &= A\,a^2 + B\,a + (M + A\,h)\,h, \end{aligned}$$

and if the root of the equation

$$A\,x^2 + B\,x = -M$$

is called the *quadratic root* of $-M$, and $-M$ *the quadratic power* of its root, *the rule for extracting its root is the same as that for extracting the square root in art.* 189, *except that* QUADRATIC *must be substituted for* SQUARE, *the divisor is, in each case,*

$$2\,A\,a + B$$

instead of 2 a, *the addition to the divisor before mul-*

tiplication is A h *instead of* h, *and the division into periods is useless.*

233. EXAMPLES.

1. Solve the equation

$$3x^2 + 5x = 1491.$$

Solution. The First Method of art. 230. Multiply by 3 and the product is

$$9x^2 + 15x = 4473.$$

The square completed by the addition of

$$(\tfrac{15}{6})^2 = (\tfrac{5}{2})^2 = \tfrac{25}{4} = 6{\cdot}25$$

is

$$(3x + \tfrac{5}{2})^2 = 4479{\cdot}25,$$

of which the square root is

$$3x + 2{\cdot}5 = \pm 66{\cdot}927,$$

whence

$$3x = 64{\cdot}427 \text{ or } -69{\cdot}427$$
$$x = 21{\cdot}476 \text{ or } -23{\cdot}142.$$

The Second Method of art. 232. In the second column of this form, the number at the top of the column is the root, the numbers above each line are the successive divisors, and the numbers below are the increased divisors before multiplication; and it is to be observed, that by the repetition of the increment the next divisor is obtained. We have, then, for the first root

	1491·	21·476
$3(20)^2 + 5 \times 20 =$	1300	
	191·	$125 = 6 \times 20 + 5$
$128 \times 1 =$	128	$128 = 125 + 3 \times 1$
	63	$131 = 6 \times 128 + 3 \times 1$
$132{\cdot}2 \times {\cdot}4 =$	52·88	$132{\cdot}2 = 131 + 3 \times {\cdot}4$
	10·12	$133{\cdot}4 = 132{\cdot}2 + 3 \times .4$
$133{\cdot}61 \times {\cdot}07 =$	9·3527	$133{\cdot}61 = 133{\cdot}4 + 3 \times {\cdot}07$
	·7673	$133{\cdot}82 = 133{\cdot}61 + 3 \times {\cdot}07$

and for the second root

1491	$-23{\cdot}142$
1100	
391·	-115
372	-124
19·	-133
13·33	$-133{\cdot}3$
5·67	$-133{\cdot}6$
5·3488	$-133{\cdot}72$
·3212	$-133{\cdot}84$

2. Solve the equation

$$\frac{48}{x+3} = \frac{165}{x+10} - 5.$$

Solution. This equation, reduced as in arts. 112 and 118, is

$$5x^2 - 52x + 135 = 0;$$

which, multiplied by 5, becomes

$$25x^2 - 260x = -675.$$

Completing the square, we have

$$25x^2 - 260x + 676 = 676 - 675 = 1,$$

the square root of which is

$$5x - 26 = \pm 1;$$

hence

$$x = \tfrac{1}{5}(26 \pm 1) = 5\tfrac{2}{5} \text{ or } = 5.$$

3. Solve the equation

$$\sqrt{(2x+7)} + \sqrt{(3x-18)} = \sqrt{(7x+1)}.$$

Solution. This equation, being freed from radical signs, as in art. 204, becomes

$$5x^2 - 27x - 162 = 0;$$

the roots of which are

$$x = 9, \text{ or } x = -3\tfrac{3}{5}.$$

4. Solve the two equations

$$(y^2+6)y+16(x-4)=\frac{(x^2-x-5)y^2+48}{x-2},$$

$$(y-5)y^2+40x^2-100=\frac{(10x^2-66x+62)y+60x}{x-2}.$$

Solution. If we proceed to eliminate y between these two equations, by the process of art. 155, the remainder of the first division is

$$(x^2-6x+5)y^2-(10x^2-60x+50)y+24x^2-144x+120,$$

in which

$$x^2-6x+5$$

is a factor of each of the coefficients of y, and y^2, and of the terms which do not contain y.

Before suppressing this factor, we must see whether, as in art. 157, it may not be equal to zero, in which case we have

$$x^2-6x+5=0,$$

the roots of which are

$$x=5, \text{ and } x=1.$$

Now if the value

$$x=5$$

is substituted in the given equations, each of them becomes

$$y^3-5y^2+6y=0,$$

which is satisfied by the value

$$y=0,$$

or, dividing by y, we have

$$y^2-5y+6=0,$$

the roots of which are

$$y=2, \text{ and } y=3.$$

But if the value

$$x=1$$

is substituted in the given equations, each of them becomes

$$y^3 - 5\,y^2 + 6\,y = 0,$$

which is the same as the preceding equation, and gives therefore the same values of y.

Having thus obtained all the roots of the given equation corresponding to

$$x^2 - 6\,x + 5 = 0,$$

we may omit this factor of the above remainder, and it becomes

$$y^2 - 10\,y + 24\,;$$

and as this does not contain x it is unnecessary to proceed farther in the elimination of y, but we may obtain the roots of the equation

$$y^2 - 10\,y + 24 = 0,$$

which are

$$y = 4, \text{ and } y = 6,$$

and substitute them in the given equation to obtain the corresponding values of x.

Thus, if the value

$$y = 6$$

is substituted in the given equations, each of them becomes

$$5\,x^2 - 48\,x + 61 = 0,$$

the roots of which are

$$x = \tfrac{1}{5}\,(24 \pm \sqrt{171}).$$

But if the value

$$y = 4$$

is substituted in the given equations, each of them becomes

$$x - 2 = 0,$$

whence

$$x = 2.$$

The answer, therefore, is

$x = 5$, or $= 1$, in either of which cases, $y = 0$, or $= 2$, or $= 3$;

or $x = \frac{1}{5}\,(24 \pm \sqrt{171})$, in which case, $y = 6$;

or $x = 2$, in which case, $y = 4$.

Examples of Equations of the Second Degree.

7. Solve the equation

$$x^2 + 8x = 209.$$

Ans. $x = 11$, or -19.

8. Solve the equation

$$4x^2 - 9x = 5x^2 - 255\tfrac{3}{4} - 8x.$$

Ans. $x = 15\tfrac{1}{2}$, or $= -16\tfrac{1}{2}$.

9. Solve the equation

$$\frac{x}{x+60} = \frac{7}{3x-5}.$$

Ans. $x = 14$, or $= -10$.

10. Solve the equation

$$\frac{8x}{x+2} - 6 = \frac{20}{3x}.$$

Ans. $x = 10$, or $= -\tfrac{2}{3}$.

11. Solve the equation

$$\frac{2x+3}{10-x} = \frac{2x}{25-3x} - 6\tfrac{1}{2}.$$

Ans. $x = 13\tfrac{22}{31}$, or $= 8$.

12. Solve the equation

$$3\sqrt{(112 - 8x)} = 19 + \sqrt{(3x + 7)}.$$

Ans. $x = 6$, or $= 11{\cdot}8369$.

13. Solve the equation

$$x^2 + 2x = 10.$$

Ans. $x = 2{\cdot}3166$, or $= -4{\cdot}3166$

14. Solve the equation

$$x^2 + 5x = 10.$$

Ans. $x = 1{\cdot}531$, or $= -6{\cdot}531$.

15. Solve the equation

$$x^2 - 9x = -10.$$

Ans. $x = 7{\cdot}7015$, or $= 1{\cdot}2984$.

16. Solve the equation

$$2x^2 + x = 11.$$

Ans. $x = 2{\cdot}108$, or $= -2{\cdot}608$.

17. Solve the equation

$$10x^2 - 14x = -3.$$

Ans. $x = 1{\cdot}1359$, or $-{\cdot}2641$.

18. Solve the two equations

$$2x + 3y = 118,$$
$$5x^2 - 7y^2 = 4333.$$

Ans. $x = 35$, and $y = 16$,
or $x = -229\tfrac{6}{17}$, and $y = 192\tfrac{4}{17}$.

19. Solve the two equations

$$x^2 - yx + 9 = 0,$$
$$y^2x^2 - y^3x + 144y - 540 = 0.$$

Ans. $y = 6$, and $x = 3$;
or $y = 10$, and $x = 9$, or $= 1$.

20. Solve the three equations

$$xyz = 105,$$
$$x + y + z = 7,$$
$$y^2 + xy - 7y - x + 22 = 0.$$

Ans. $x = 15$; $y = -1$, $z = -7$;
or $x = 15$, $y = -7$, $z = -1$;
or $x = 7$, $y = +\sqrt{15}$, $z = -\sqrt{15}$;
or $x = 7$, $y = -\sqrt{15}$, $z = +\sqrt{15}$.

21. What two numbers are they, whose sum is 32, and product 240? *Ans.* 12 and 20.

22. What two numbers are they, whose sum is a, and product b?

Ans. $\frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - b)}$, and $\frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - b)}$.

In what case would the values of these unknown quantities be imaginary?

Ans. When we have

$$b > \frac{1}{4} a^2,$$

that is,

$$b > (\tfrac{1}{2} a)^2;$$

that is, the product of two numbers cannot be greater than the square of half their sum.

23. What two numbers are they, whose difference is 5, and product 24? *Ans.* 8 and 3; or -3 and -8.

24. What two numbers are they, whose difference is a, and product b?

Ans. $\frac{1}{2} a \pm \sqrt{(b + \frac{1}{4} a^2)}$, and $-\frac{1}{2} a \pm \sqrt{(b + \frac{1}{4} a^2)}$.

25. Find a number, whose square exceeds it by 306.

Ans. 18, or -17.

26. A person being asked his age, answered, "My mother was 20 years old when I was born, and her age multiplied by mine, exceeds our united ages by 2500." What was his age? *Ans.* 42.

27. A person buys some pieces of cloth, at equal prices, for \$ 60. Had he got three more pieces for the same sum, each piece would have cost him \$ 1 less. How many pieces did he buy? *Ans.* 12.

28. A person dies, leaving children, and a fortune of \$ 46800, which, by the will, is to be divided equally among them. It happens, however, that immediately after the death of the father, two of his children also die. If, in consequence of this, each remaining child receives \$ 1950 more than it was entitled to by the will, how many children were there? *Ans.* 8.

29. Twenty persons, men and women, spent \$ 48 at an inn; the men \$ 24, and the women the same sum. Now, on inspecting the bill, it is found that the men have to pay

\$1 each more than the women. How many men, therefore, were there in the company? *Ans.* 8.

30. What two numbers are they, whose sum is 41, and the sum of whose squares is 901? *Ans.* 15 and 26.

31. What two numbers are they, whose sum is a, and the sum of whose squares is b?

Ans. $\frac{1}{2}a + \frac{1}{2}\sqrt{(2b - a^2)}$, and $\frac{1}{2}a - \frac{1}{2}(2\sqrt{b} - a^2)$.

In what case would the values of these unknown quantities be imaginary?

Ans. When we have

$$a^2 > 2b;$$

that is, the square of the sum of two numbers cannot be greater than twice the sum of their squares.

32. What two numbers are they, whose difference is 8, and the sum of whose squares is 544?

Ans. 12 and 20; or -12 and -20.

33. What two numbers are they, whose difference is a, and the sum of whose squares is b?

Ans. $\frac{1}{2}a \pm \frac{1}{2}\sqrt{(2b - a^2)}$, and $-\frac{1}{2}a \pm \frac{1}{2}\sqrt{(2b - a^2)}$.

In what case would the values of these unknown quantities be imaginary?

Ans. When we have

$$a^2 > 2b;$$

that is, the square of the difference of two numbers cannot be greater than twice the sum of their squares.

34. Divide the number 39 into two parts, such that the sum of their cubes may be 17199. *Ans.* 15 and 24.

35. A person being asked about his yearly income, answered, "My income is such, that if I add \$1578 to it, and also subtract \$142 from it, and extract the cube roots

of the numbers thus obtained, the difference between the roots is 10." What was his income? *Ans.* \$150.

36. Find two numbers whose difference added to the difference of their squares, makes 150, and whose sum added to the sum of their squares is 330.

Ans. The one is 15, or -16; the other is 9, or -10.

37. What two numbers are they, whose sum, product, and difference of their squares, are all equal to each other?

Ans. $\frac{1}{2}(3 \pm \sqrt{5})$, and $\frac{1}{2}(1 \pm \sqrt{5})$.

38. Find a number consisting of three digits, such, that the sum of the squares of the digits, without considering their position, may be 104; but the square of the middle digit exceeds twice the product of the other two by 4; farther, if 594 be subtracted from the number sought, the three digits are inverted. *Ans.* 862.

234. *Corollary.* The preceding method is not only applicable to equations of the second degree, but to all equations of the form

$$A x^{2n} + B x^n + M = 0,$$

in which there are two terms multiplied by different powers of x, the highest exponent being the double of the lowest; and n may be either integral or fractional.

235. EXAMPLES.

1. Solve the equation

$$A x^{2n} + B x^n + M = 0$$

Solution. If the square is completed, as in the preceding article, and the square root extracted, the result is

$$A x^n + \tfrac{1}{2} B = \pm \sqrt{(-A M + \tfrac{1}{4} B^2)};$$

Examples of Quadratic Equations higher than the Second Degree.

from which we obtain, by art. 224,

$$x = \left(\frac{-\frac{1}{2}B \pm \sqrt{(-AM + \frac{1}{4}B^2)}}{A}\right)^{\frac{1}{n}}.$$

2. Solve the equation

$$x^4 - 74\,x^2 = -1225.$$

Ans. $x = \pm 5$, or $= \pm 7$.

3. Solve the equation

$$3\,x^6 + 42\,x^3 = 3321.$$

Ans. $x = 3$, or $= -\sqrt[3]{41}$.

4. Solve the equation

$$5\sqrt[4]{x} - \sqrt{x} = 6.$$

Ans. $x = 16$, or 81.

5. Solve the equation

$$(x + 12)^{\frac{1}{2}} + (x + 12)^{\frac{1}{4}} = 6.$$

Ans. $x = 4$, or 69.

6. Solve the equation

$$x + 16 - 7\,(x + 16)^{\frac{1}{2}} = 10 - 4\,(x + 16)^{\frac{1}{2}}.$$

Ans. $x = 9$, or -12.

7. Solve the equation

$$x^{\frac{3}{2}} - x = 2\,x^{\frac{1}{2}}.$$

Ans. $x = 0$, or 1, or 4.

8. Solve the equation

$$x^3 - x^{\frac{3}{2}} = 56.$$

Ans. $x = 4$, or $(-7)^{\frac{2}{3}}$.

9. Solve the equation

$$x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756.$$

Ans. $x = 243$, or $(-28)^{\frac{5}{3}}$.

Examples of Substitution of Unknown Quantities.

10. Solve the equation

$$(x^2+5)^2-4\,x^2=160.$$

Ans. $x=3$, or $\sqrt{-15}$.

11. What two numbers are they, whose product is 255, and the sum of whose squares is 514?

Ans. 15 and 17, or -15 and -17.

12. What two numbers are they, whose product is a, and the sum of whose squares is b?

Ans. $\pm\sqrt{[\frac{1}{2}\,b+\sqrt{(\frac{1}{4}\,b^2-a^2)}]}$,
and $\pm\sqrt{[\frac{1}{2}\,b-\sqrt{(\frac{1}{4}\,b^2-a^2)}]}$.

13. What number exceeds its square root by 20?

Ans. 25.

14. What number is it, the excess of whose square above its square root is equal to 56 divided by the number?

Ans. 4 or $\sqrt[3]{49}$.

236. There are equations of higher degrees, which can be reduced to equations of the second degree by introducing other unknown quantities instead of those contained in them. Thus if the same algebraic expression is involved in different ways, it will often be found successful to consider this expression as the unknown quantity.

237. EXAMPLES.

1. Solve the two equations

$$(x^2-23\,y)^3+(x^2-23\,y)^2+(x^2-23\,y)\,(x-2\,y)=18,$$
$$(x^2-23\,y)^2+(x-2\,y)=7.$$

Solution. Consider

$$(x^2 - 23\,y),\text{ and }(x - 2\,y),$$

as the unknown quantities, making

$$x' = x^2 - 23\,y,$$
$$y' = x - 2\,y;$$

and the equations become

$$x'^3 + x'^2 + x'\,y' = 18,$$
$$x'^2 + y' = 7.$$

Hence, by the elimination of y', we have

$$x'^2 + 7\,x' = 18,$$

and, therefore,

$$x' = 2,\text{ or } = -9;$$

and the corresponding values of y' are

$$y' = 3,\text{ or } = -74;$$

that is,

$$x^2 - 23\,y = 2,\text{ or } = -9,$$
$$x - 2\,y = 3,\text{ or } = -74.$$

The solution of these equations gives

$$x = 5,\quad y = 1;$$

or,

$$x = 6\tfrac{1}{2},\ y = 1\tfrac{3}{4};$$

or,

$$x = \tfrac{1}{4}(23 \pm \sqrt{14001}),\ y = \tfrac{1}{8}(319 \pm \sqrt{14001})$$

2. Solve the equation

$$x + (x + 6)^{\frac{1}{2}} = 2 + 3\,(x + 6)^{\frac{1}{2}}.$$

Ans. $x = 10$, or -2.

3. Solve the two equations

$$(x + y) + (x + y)^{\frac{1}{2}} = 12,$$
$$x^3 + y^3 = 189.$$

Ans. $x = 5$, or $= 4$; $y = 4$, or $= 5$.

238. *Corollary.* When there are two unknown quantities which enter symmetrically into the given equation, the solution is often simplified by substituting for them two other unknown quantities, one of which is their product and the other their sum.

239. EXAMPLES.

1. Find two numbers whose sum is 5, and the sum of whose fifth powers is 275.

Solution. Let the numbers be x and y, represent their product by p, and we have

$$x + y = 5,$$
$$x^5 + y^5 = 275.$$

But we also have

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$
$$= x^5 + y^5 + 5xy(x^3 + y^3) + 10x^2y^2(x + y);$$

and

$$x^3 + y^3 = (x + y)^3 - 3x^2y - 3xy^2$$
$$= (x + y)^3 - 3xy(x + y)$$
$$= 125 - 15p.$$

Hence

$$(x + y)^5 = 275 + 5p(125 - 15p) + 10p^2 \times 5 = 5^5;$$

or, by reduction,

$$p^2 - 25p = -114,$$
$$p = 19, \text{ or } = 6;$$

and

$$x = 2, \text{ or } = 3, \text{ or } = \tfrac{1}{2}(5 \pm \sqrt{-51}),$$
$$y = 3, \text{ or } = 2, \text{ or } = \tfrac{1}{2}(5 \mp \sqrt{-51}).$$

2. Solve the two equations

$$(x - y)(x^2 - y^2) = 7,$$
$$(x + y)(x^2 + y^2) = 175.$$

Solution. These equations become, by development,

$$x^3 - x^2 y - x y^2 + y^3 = 7,$$
$$x^3 + x^2 y + x y^2 + y^3 = 175;$$

and, by the substitution of

$$x + y = s,$$
$$x y = p;$$

they still farther become,

$$s^3 - 4 s p = 7,$$
$$s^3 - 2 s p = 175.$$

If we eliminate p we have

$$s^3 = 343,$$

whence

$$s = 7;$$

and this value gives, by substitution,

$$343 - 14 p = 175,$$
$$p = 12.$$

Hence

$$x = 3, \text{ or } = 4;$$
$$y = 4, \text{ or } = 3.$$

3. Solve the two equations

$$x + y = x y$$
$$x + y + x^2 + y^2 = 12.$$

Ans. $x = 2$, or $= \frac{1}{2}(-3 \pm \sqrt{21})$;
$y = 2$, or $= \frac{1}{2}(-3 \mp \sqrt{21})$.

4. Solve the two equations

$$x^3 + y^3 = 189,$$
$$x^2 y + x y^2 = 180,$$

Ans. $x = 4$, or $= 5$;
$y = 5$, or $= 4$.

5. Solve the two equations

$$x^2 + y^2 = 5,$$
$$x y = 2.$$

Ans. $x = \pm 2$, or $= \pm 1$;
$y = \pm 1$, or $= \pm 2$.

6. Solve the two equations

$$x^2 y + x y^2 = 6,$$
$$x^3 y^2 + x^2 y^3 = 12.$$

Ans. $x = 1$, or $= 2$;
$y = 2$, or $= 1$.

7. Solve the two equations

$$4 x y = 96 - x^2 y^2,$$
$$x + y = 6.$$

Ans. $x = 2$, or 4, or $3 \pm \surd 21$;
$y = 4$, or 2, or $3 \mp \surd 21$.

8. Find two numbers such, that their sum and product may together be 34, and the sum of their squares may exceed the sum of the numbers themselves by 42.

Ans. 4 and 6;
or $\frac{1}{2}(-11 + \surd -59,)$ and $\frac{1}{2}(-11 - \surd -59)$.

9. What two numbers are they, whose sum is 3, and the sum of whose fourth powers is 17?

Ans. 2 and 1;
or $\frac{1}{2}(3 + \surd -55)$, and $\frac{1}{2}(3 - \surd -55)$.

10. What two numbers are they, whose product is 3, and the sum of whose fourth powers is 82?

Ans. ± 1, and ± 3;
or $\pm \surd - 1$, and $\mp \surd - 9$

240. *Corollary.* In many cases, in which two unknown quantities enter into the given equations symmetrically except in regard to their signs, the solution is simplified by substituting for them two other unknown quantities, one of which is their difference, and the other is their sum or their product.

241. EXAMPLES.

1. Solve the two equations

$$(x-y)(x^2+y^2)=13,$$
$$(x-y)\,x\,y=6.$$

Solution. These equations become, by the substitution of

$$x-y=t,$$
$$x\,y=p;$$
$$t\,(t^2+2\,p)=13,$$
$$t\,p=6.$$

By the elimination of p, we have

$$t^3=1,$$
$$t=1;$$

whence we find

$$p=6,$$

and

$$x=3,\text{ or }=-2;$$
$$y=2,\text{ or }=-3.$$

2. Solve the two equations

$$x^2-y^2=7,$$
$$x^3+y^3=91\,(x-y).$$

Solution. These equations become, by the substitution of

$$x+y=s,$$
$$x-y=t;$$
$$s\,t=7,$$
$$\tfrac{1}{4}\,(s^3+3\,s\,t^2)=91\,t.$$

Hence, by the elimination of t, we have

$$s^4-2401=0,$$

and

$$s = \sqrt[4]{2401} = \pm 7, \text{ or } = \pm 7\sqrt{-1};$$
$$t = \pm 1, \text{ or } = \mp \sqrt{-1}.$$
$$x = \pm 4, \text{ or } = \pm 3\sqrt{-1};$$
$$y = \pm 3, \text{ or } = \pm 4\sqrt{-1}.$$

3. Solve the two equations

$$x^3 - y^3 = 7,$$
$$(x^2 + y^2)(x - y) - (x - y)xy = 3.$$

Ans. $x = 2$, or $= -1$;
$y = 1$, or $= -2$.

4. Solve the two equations

$$x^3 - y^3 = 215,$$
$$x^2 + xy + y^2 = 43.$$

Ans. $x = 6$, or $= -1$;
$y = 1$, or $= -6$.

5. Solve the two equations

$$x^3 y - x^2 y^2 + x y^3 = 156,$$
$$xy(x^3 - y^3) - 2x^2y^2(x - y) + (x - y)^2 = 157.$$

Ans. $x = 4$, or $= -3$, or $= \frac{1}{2}(\ \ 1 \pm \sqrt{-51})$;
$y = 3$, or $= +4$, or $= \frac{1}{2}(-1 \pm \sqrt{-51})$.

or $x = \pm \frac{1}{2}\sqrt{\left(-\overline{157}^2 \pm 2\sqrt{(624 + \overline{157}^4)}\right)} - 78\frac{1}{2}$,
$y = \pm \frac{1}{2}\sqrt{\left(-\overline{157}^2 \pm 2\sqrt{(624 + \overline{157}^4)}\right)} + 78\frac{1}{2}$.

6. What two numbers are they, whose difference is 1, and the difference of whose third powers is 7?

Ans. 1 and 2, or -2 and -1.

7. What two numbers are they, whose difference is 3, and the sum of whose fourth powers is 257?

Ans. 4 and 1, or -4 and -1,
or $\frac{1}{2}(\pm\sqrt{(-79)} + 3)$ and $\frac{1}{2}(\pm\sqrt{(-79)} - 3)$.

242. When the first member of one of the equations, reduced as in art. 118, is homogeneous in regard to two unknown quantities, the solution is often simplified by substituting for the two unknown quantities, two other unknown quantities, one of which is their quotient.

The same method of simplification can also be employed when such a homogeneous equation is readily obtained from the given equations.

243. EXAMPLES.

1. Solve the two equations

$$x^2 - 6xy + 8y^2 = 0,$$
$$x^2y + 6xy^2 + 8y^3 + (x - 2y)(y^2 - 5y + 4 = 0.$$

Solution. Retaining the unknown quantity y, introduce instead of x, the unknown quantity q, such that

$$q = \frac{x}{y},$$
$$\text{or } x = qy;$$

from which the given equations become

$$q^2y^2 - 6qy^2 + 8y^2 = 0,$$
$$q^2y^3 - 6qy^3 + 8y^3 + (qy - 2y)(y^2 - 5y + 4) = 0.$$

Both these equations are satisfied by the value of y,

$$y = 0,$$

whence

$$x = qy = 0.$$

But if we divide the first of these equations by y^2, and the second by y, we have

$$q^2 - 6q + 8 = 0,$$
$$q^2y^2 - 6qy^2 + 8y^2 + (q - 2)(y^2 - 5y + 4) = 0;$$

the first of which gives

$$q = 2, \text{ or } = 4.$$

The value of q,

$$q = 2,$$

being substituted in the other equation, reduces the first member to zero, and therefore y is indeterminate; that is, x and y may have any values whatever, with the limitation that x is the double of y.

The value of q,

$$q = 4.$$

being substituted in the other equation, gives

$$2\,(y^2 - 5\,y + 4) = 0;$$

whence

$$y = 1, \text{ or } = 4,$$

and

$$x = 4, \text{ or } = 16.$$

2. Solve the two equations

$$x^5 + x^3\,y^2 = 5,$$
$$x^5 + 4\,x\,y^4 = 65.$$

Solution. 13 times the first equation, diminished by the second equation, is

$$12\,x^5 + 13\,x^3\,y^2 - 4\,x\,y^4 = 0;$$

and, if we make

$$x = q\,y,$$

we have

$$12\,q^5\,y^5 + 13\,q^3\,y^5 - 4\,q\,y^5 = 0.$$

Which is satisfied by the value of y,

$$y = 0;$$

and this value of y, being substituted in the given equations produces

$$x^5 = 5,$$
$$x^5 = 65;$$

which are evident impossibilities, and therefore the value $y = 0$ is impossible.

Dividing, then, by y^5, we have

$$12\,q^5 + 13\,q^3 - 4\,q = 0;$$

which is satisfied by the value of q,

$$q = 0;$$

dividing by q, we have

$$12\,q^4 + 13\,q^2 - 4 = 0,$$

whence

$$q = \pm \tfrac{1}{2}, \text{ or } q = \pm \sqrt{-\tfrac{4}{3}}.$$

Now the first of the given equations becomes, by the substitution of

$$x = q\,y,$$
$$q^5\,y^5 + q^3\,y^5 = 5;$$

hence, by the substitution of the above values of q, we have

$$y = \infty,\ x = 0 \times \infty = \tfrac{0}{0} = \text{indeterminate};$$
$$\text{or } y = \pm 2,\ x = 1;$$
$$\text{or } y = \pm \sqrt[5]{\tfrac{3}{8}} \times \sqrt{-3},\ x = \sqrt[5]{20}.$$

3. Solve the two equations

$$81\,x^4 + 9\,x^2\,y^2 = 20\,y^4,$$
$$(y^2 - y)^2 + (3\,x\,y + 2\,y)^2 - 9\,x^2(2\,y + 3) - 12\,y\,(x + 2\,y) = 0.$$

Ans. $x = 0$, and $y = 0$;
or $x = 2$, and $y = 3$;
or $x = -1\frac{11}{15}$, and $y = -2\frac{3}{5}$;
or $x = -3\frac{1}{15}$, and $y = 4\frac{3}{5}$;
or $x = \frac{2}{3}$, and $y = -1$;
or $x = \frac{1}{3}(-5 \pm \sqrt{-5},)$ and $y = 1 \pm \sqrt{-5}$;
or $x = \pm \frac{1}{3}\sqrt{-5}$, and $y = 1$.

4. Solve the two equations

$$x^3 + 2\,x\,y^2 = 3,$$
$$x\,y^2 + 2\,x^2\,y = 3.$$

Ans. $x = 1$, and $y = 1$.

5. What two numbers are they, twice the sum of whose squares is 5 times their product, and the sum of whose sixth powers is 65. *Ans.* 2 and 1, or -2 and -1.

6. What two numbers are they, the difference of whose fourth powers is 65, and the square of the sum of whose squares is 169. *Ans.* ± 2, and ± 3.

CHAPTER VII.

PROGRESSIONS.

SECTION I.

Arithmetical Progression.

244. An *Arithmetical Progression*, or a *progression by differences*, is a series of terms or quantities which continually increase or decrease by a constant quantity.

This constant increment or decrement is called the *common difference* of the progression.

Throughout this section the following notation will be retained. We shall use

$a =$ the first term of the progression,
$l =$ the last term,
$r =$ the common difference,
$n =$ the number of terms,
$S =$ the sum of all the terms.

245. *Problem. To find the last term of an arithmetical progression when its first term, common difference, and number of terms are known.*

Solution. In this case a, r, and n, are supposed to be known, and l is to be found. Now the successive terms of the series if it is increasing are

$$a, a+r, a+2r, a+3r, a+4r, \text{\&c.};$$

so that the nth term is obviously

$$l = a + (n - 1) r.$$

But if the series is decreasing, the last term must be

$$l = a - (n - 1) r.$$

Both these cases are, however, included in one, if we suppose r to be negative when the series is decreasing.

246. *Corollary.* In like manner any other term, such as the mth, is

$$a + (m - 1) r.$$

247. *Corollary.* By writing the series in an inverted order, beginning with the last term, a new series is found, of which the first term is l, and the common difference $-r$. Hence the mth term of this series, that is, the mth term counting from the last of the given series, is

$$l - (m - 1) r.$$

248. *Corollary.* The sum of the mth term and of the mth term from the last is, therefore,

$$[a + (m - 1) r] + [l - (m - 1) r] = a + l;$$

that is, *the sum of any two terms, taken at equal distances from the two extremes of an arithmetical series, is equal to the sum of the two extremes.*

249. *Problem. To find the sum of an arithmetical progression when its first term, last term, and number of terms are known.*

Solution. In this case, a, l, and n are supposed to be known, and S is to be found.

To find the Sum of the Progression.

Suppose the terms of the series to be written as follows first in the regular order, and then in an inverted order:

$$a,\ b,\ c,\ \ldots\ldots\ i,\ k,\ l;$$
$$l,\ k,\ i,\ \ldots\ldots\ c,\ b,\ a.$$

The sum of the terms of each of these progressions being S, the sum of both of them must be $2S$, that is,

$$2S=(a+l)+(b+k)+(c+i)\ldots+(i+c)+(k+b)+(l+a).$$

But by the preceding corollary, we have

$$a+l=b+k=c+i=\&c.$$

Hence $2S$ is equal to as many times $(a+l)$ as there are terms in the series, that is,

$$2S=(a+l)n;$$
$$\text{or } S=\tfrac{1}{2}(a+l)n;$$

that is, *the sum of a progression is equal to half the sum of the two extremes, multiplied by the number of terms.*

250. *Corollary.* From the equations

$$l=a+(n-1)r,$$
$$S=\tfrac{1}{2}(a+l)n;$$

either two of the quantities a, l, r, n, and S can be determined when the other three are known.

251. EXAMPLES.

1. Find the 100th term of the series 2, 9, 16, &c.

Ans. 695.

2. Find the sum of the preceding series.

Ans. 34850.

3. Find S, when a, r, and n are known.

Ans. $S=\tfrac{1}{2}[2a+(n-1)r]n$.

Examples in Progression.

4. Find n and S, when a, l, and r are known.

$$Ans.\ n = \frac{l-a}{r} + 1;$$

$$S = \frac{l^2 - a^2}{2r} + \tfrac{1}{2}(a+l).$$

5. Find the number and sum of terms of the series of which the first term is 6, the last term 796, and the common difference 10.

Ans. The number of terms $= 80$,
the sum $= 32080$.

6. Find r, when a, l, and n are known.

$$Ans.\ r = \frac{l-a}{n-1}.$$

7. Find the common difference and sum of the series, of which the first term is 75, and the last term 15, and the number of terms 13.

Ans. The common difference $= -5$,
the sum $= 585$.

8. Find r and n, when a, l, and S are known.

$$Ans.\ n = \frac{2S}{a+l},$$

$$r = \frac{l^2 - a^2}{2S - (a+l)}.$$

9. Find the common difference and number of terms of a series, of which the first term is 2, and the last term 345, and the sum 8675.

Ans. The number of terms $= 50$,
the common difference $= 7$.

10. Find l and n, when a, r, and S are known.

$$Ans.\ n = \frac{\sqrt{[2rS + (a - \frac{1}{2}r)^2]} - (a - \frac{1}{2}r)}{r},$$

$$l = \sqrt{[2rS + (a - \tfrac{1}{2}r)^2]} - \tfrac{1}{2}r.$$

11. Find the last term and number of terms of a series, of which the first term is 3, the common difference 4, and the sum of the terms 105.

Ans. The last term $= 27$,
the number of terms $= 7$.

12. Find a and n, when l, r, and S are known.

$$Ans.\ n = \frac{l + \frac{1}{2}r \mp \sqrt{[(l + \frac{1}{2}r)^2 - 2rS]}}{r}.$$

$$a = \pm\sqrt{[(l + \tfrac{1}{2}r)^2 - 2rS]} + \tfrac{1}{2}r.$$

13. Find the first term and the number of terms of a series, of which the last term is 13, the common difference 3, and the sum of the series 35.

Ans. The first term $= 1$,
the number of terms $= 5$.

14. Find l and r, when a, n, and S are known.

$$Ans.\ l = \frac{2S}{n} - a,$$

$$r = \frac{2(S - an)}{n(n-1)}.$$

15. Find the last term and common difference of a series, of which the first term is $\frac{2}{3}$, the number of terms 12, and the sum 100.

Ans. The last term $= 16$,
the common difference $= 1\frac{13}{33}$.

16. Find a and r, when l, n, and S are known.

$$Ans.\ a = \frac{2S}{n} - l,$$

$$r = \frac{2(ln - S)}{n(n-1)}.$$

17. Find the first term and common difference of a series, of which the last term is 50, the number of terms 20, and the sum 600.

Ans. The first term $= 10$,
the common difference $= 2\frac{2}{19}$.

18. Find a and S, when l, r, and n are known.

Ans. $a = l - (n - 1)\,r$,
$S = \frac{1}{2}\,[2\,l - (n - 1)\,r]\,n$.

19. Find the first term and sum of the terms of a series, of which the last term is 100, the common difference $\frac{1}{2}$, and the number of terms 51.

Ans. The first term $= 75$,
the sum of the terms $= 4462\frac{1}{2}$.

20. Find a and l, when r, n, and S are known.

$$Ans.\ a = \frac{S}{n} - \frac{1}{2}\,(n - 1)\,r,$$

$$l = \frac{S}{n} + \frac{1}{2}\,(n - 1)\,r.$$

21. Find the first and last terms of a series, of which the common difference is 5, the number of terms 6, and the sum 321.

Ans. The first term $= 41$,
the last term $= 66$.

22. Find the sum of the natural series of numbers 1, 2, 3, &c. up to n terms.

Ans. $\frac{1}{2}\,n\,(n + 1)$.

23. Find the sum of the natural series of numbers from 1 to 100.

Ans. 5050.

24. Find the sum of the odd numbers 1, 3, 5, &c. up to n terms.

Ans. n^2

Examples in Progression.

25. Find the sum of the odd numbers from 1 to 99.

Ans. 2500.

26. Find the sum of the even numbers 2, 4, 6, &c. up to n terms.

Ans. $n(n+1)$.

27. Find the sum of the even numbers from 2 to 100.

Ans. 2550.

28. One hundred stones being placed on the ground, in a straight line, at a distance of 2 yards from each other; how far will a person travel, who shall bring them one by one to a basket, placed at 2 yards from the first stone?

Ans. 11 miles, 840 yards.

29. We know, from natural philosophy, that, a body which falls in a vacuum, passes, in the first second of its fall, through a space of $16\frac{1}{12}$ feet, but in each succeeding second, $32\frac{1}{6}$ feet more than in the immediately preceding one. Now, if a body has been falling 20 seconds, how many feet will it have fallen the last second? and how many in the whole time?

Ans. $627\frac{1}{4}$ feet in the last second, and $6433\frac{1}{3}$ feet in the whole time.

30. In a foundery, a person saw 15 rows of cannon-balls placed one above another, and asked a bombardier how many balls there were in the lowest row. "You may easily calculate that," answered the bombardier. "In all these rows together, there are 4200 balls, and each row, from the first to the last, contains 20 balls less than the one immediately below it." How many balls, therefore, were there in the lowest row?

Ans. 420.

252. The *arithmetical mean* between several

quantities is the quotient of their sum divided by their number.

Thus the arithmetical mean between the two quantities a and b is half their sum, or $\frac{1}{2}(a+b)$; that between the four quantities 1, 7, 11, 5 is 6.

253. *Problem.* To find the arithmetical mean between the terms of an arithmetical progression.

Scholium. It is, by the preceding definition

$$\frac{S}{n},$$

or, since

$$S = \tfrac{1}{2} n (a + l),$$

it is

$$\tfrac{1}{2}(a + l);$$

that is, ***half the sum of the extremes***, and also, by art. 248, ***half the sum of any two terms at equal distances from the extremes.***

254. *Problem.* To find the first and last terms of a progression of which the arithmetical mean, the number of terms, and the common difference are known.

Solution. If we denote the arithmetical mean by M, we have

$$M = \frac{S}{n} = \tfrac{1}{2}(a + l);$$

which, substituted in the result of example 20, in art. 251, gives

$$a = M - \tfrac{1}{2}(n - 1)\,r,$$
$$l = M + \tfrac{1}{2}(n - 1)\,r.$$

255. *Scholium.* In very many of the problems involving arithmetical progression, it is convenient

to use for one of the unknown quantities the arithmetical mean.

256. EXAMPLES.

1. Find five numbers in arithmetical progression whose sum is 25, and whose continued product is 945.

Solution. Denote the arithmetical mean by M, and the common difference by r, and we have, by art. 254,

$$M = \frac{25}{5} = 5;$$

and

$$\begin{aligned}
&\text{the first term} &&= M - 2r &&= 5 - 2r,\\
&\text{the second term} &&= M - r &&= 5 - r,\\
&\text{the third term} &&= M &&= 5,\\
&\text{the fourth term} &&= M + r &&= 5 + r,\\
&\text{the fifth term} &&= M + 2r &&= 5 + 2r;
\end{aligned}$$

and the continued product of these terms is

$$(5-2r)(5-r)5(5+r)(5+2r) = 3125 - 625r^2 + 20r^4 = 945.$$

Hence we find

$$r = \pm 2, \text{ or } = \pm \sqrt{54\tfrac{1}{2}};$$

and the only rational series satisfying the condition is, therefore, 1, 3, 5, 7, 9.

2. Find four numbers in arithmetical progression whose sum is 32, and the sum of whose squares is 276.

Ans. 5, 7, 9, 11.

3. A traveller sets out for a certain place, and travels 1 mile the first day, 2 the second, and so on. In five days afterwards another sets out, and travels 12 miles a day. How long and how far must he travel to overtake the first?

Ans. 3 days and 36 miles.

4. Find four numbers in arithmetical progression whose sum is 28, and continued product 585.

Ans. 1, 5, 9, 13.

5. The sum of the squares of the first and last of four numbers in arithmetical progression is 200, and the sum of the squares of the second and third is 136; find them.

Ans. 2, 6, 10, 14.

6. Eighteen numbers in arithmetical progression are such, that the sum of the two mean terms is $31\frac{1}{2}$, and the product of the extreme terms is $85\frac{1}{2}$. Find the first term and the common difference.

Ans. The first term is 3,
the common difference is $1\frac{1}{2}$.

SECTION II.

Geometrical Progression.

257. A *Geometrical Progression*, or a *progression by quotients*, is a series of terms which increase or decrease by a constant ratio.

a, l, n, and S will be used in this section as in the last, to denote respectively the first term, the last term, the number of terms, and the sum of the terms; and r will be used to denote the constant ratio.

258. ***Problem.*** ***To find the last term of a geometrical progression when its first term, ratio, and number of terms are known.***

Solution. In this case a, r, and n are given, to find l Now the terms of the series are as follows:

$$a,\ ar,\ ar^2,\ ar^3 \ldots \&c. \ldots ar^{n-1},$$

so that, the last or nth term is

$$l = ar^{n-1};$$

that is, *the last term is equal to the product of the first term by that power of the ratio whose exponent is one less than the number of terms.*

259. *Problem. To find the sum of a geometrical progression, of which the first term, the ratio, and the number of terms are known.*

Solution. We have

$$S = a + ar + ar^2 + \&c. \ldots + ar^{n-2} + ar^{n-1}.$$

If we multiply all the terms of this equation by r, we have

$$rS = ar + ar^2 + ar^3 + \&c. \ldots + ar^{n-1} + ar^{+n};$$

from which, subtracting the former equation, and striking out the terms which cancel, we have

$$rS - S = ar^n - a,$$

or

$$(r-1)\,S = ar^n - a = a\,(r^n - 1);$$

whence

$$S = \frac{ar^n - a}{r-1} = \frac{a\,(r^n - 1)}{r-1}.$$

Hence, *to find the sum, multiply the first term by the difference between unity and that power of the ratio whose exponent is equal to the number of terms, and divide the product by the difference between unity and the ratio.*

Examples in Geometrical Progression.

260. *Corollary*. The two equations

$$l = a\,r^{n-1}$$
$$(r-1)\,S = a\,(r^n - 1)$$

give the means of determining either two of the quantities a, l, r, n, and S, when the other three are known.

But it must be observed, that, since n is an exponent, it can only be determined by the solution of an exponential equation.

261. EXAMPLES

1. Find the 8th term and the sum of the first 8 terms of the progression 2, 6, 18, &c., of which the ratio is 3.

Ans. The 8th term is 4374, the sum is 6560.

2. Find the 12th term and the sum of the first 12 terms of the series 64, 16, 4, 1, $\frac{1}{4}$, &c., of which the ratio is $\frac{1}{4}$.

Ans. The 12th term is $\frac{1}{65536}$, the sum is $85\frac{65535}{196608}$.

3. Find S, when a, l, and r are known.

$$Ans.\ S = \frac{r\,l - a}{r-1}.$$

4. Find the sum of the geometrical progression of which the first term is 7, the ratio $\frac{1}{2}$, and the last term $1\frac{3}{4}$.

Ans. $12\frac{1}{4}$.

5. Find r and S, when a, l, and n are known.

$$Ans.\ r = \sqrt[n-1]{\frac{l}{a}},\ S = \frac{l\sqrt[n-1]{\frac{l}{a}} - a}{\sqrt[n-1]{\frac{l}{a}} - 1} = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$$

Examples in Geometrical Progression.

6. Find the ratio and sum of the series of which the first term is 160, the last term 38880, and the number of terms 6.

Ans. The ratio is 3,
the sum is 58240.

7. Find r, when a, l, and S are known.

$$Ans.\ r = \frac{S - a}{S - l}.$$

8. Find the ratio of the series of which the first term is 1620, the last term 20, and the sum 2420.

Ans. $\frac{1}{3}$.

9. Find a and S, when l, r, and n are known.

$$Ans.\ a = \frac{l}{r^{n-1}},$$

$$S = \frac{l\,(r^{n}-1)}{r^{n} - r^{n-1}}.$$

10. Find the first term and sum of the series of which the last term is 1, the ratio $\frac{1}{2}$, and the number of terms 5.

Ans. The first term is 16,
the sum is 31.

11. Find l, when a, r, and S are known.

$$Ans.\ l = S - \frac{S - a}{r}.$$

12. Find the last term of the series of which the first term is 5, the ratio $\frac{1}{5}$, and the sum $6\frac{6}{25}$.

Ans. $\frac{1}{25}$.

13. Find a, when l, r, and S are known.

$$Ans.\ a = S - (S - l)\,r.$$

14. Find the first term of the series of which the last term is $\frac{1}{25}$, the ratio $\frac{1}{5}$, and the sum $6\frac{6}{25}$

Ans. 5.

15. Find a and l, when r, n, and S are known.

$$Ans.\ a = \frac{(r-1)\,S}{r^n - 1},$$

$$l = \frac{(r^n - r^{n-1})\,S}{r^n - 1}.$$

16. Find the first and last terms of the series of which the ratio is 2, the number of terms 12, and the sum 4095.

Ans. The first term is 1,
the last term 2048.

262. An *infinite decreasing* geometrical progression is one in which the ratio is less than unity, and the number of terms infinite.

263. *Problem. To find the last term and the sum of the terms of an infinite decreasing geometrical progression, of which the first term and the ratio are known.*

Solution. Since r is less than unity, we may denote it by a fraction, of which the numerator is 1, and the denominator r' is greater than unity; and we have

$$r = \frac{1}{r'},$$

$$r^{\infty} = \frac{1}{r'^{\infty}} = \frac{1}{\infty} = 0.$$

Since, then, the number of terms is infinite, the formulas for the last term and the sum become

$$l = a\,r^{n-1} = a \times 0 = 0,$$

$$S = \frac{r\,l - a}{r-1} = \frac{-a}{r-1},$$

$$= \frac{a}{1-r} = \frac{a\,r'}{r'-1},$$

that is, *the last term is zero, and the sum is found by dividing the first term by the difference between unity and the ratio.*

264. *Corollary.* From the equation

$$S = \frac{a}{1 - r},$$

either of the quantities a, r, and S may be found, when the other two are known.

265. EXAMPLES.

1. Find the sum of the infinite progression, of which the first term is 1, and the ratio $\frac{1}{2}$.

Ans. 2.

2. Find the sum of the infinite progression of which the first term is 0·7, and the ratio 0·1.

Ans. $\frac{7}{9}$.

3. Find r, in an infinite progression, when a and S are known.

Ans. $r = 1 - \dfrac{a}{S}$.

4. Find the ratio of an infinite progression, of which the first term is 17, and the sum 18.

Ans. $\frac{1}{18}$.

5. Find a, in an infinite progression, when r and S are known.

Ans. $a = S(1 - r)$.

6. Find the first term of an infinite progression, of which the ratio is $\frac{2}{3}$, and the sum 6.

Ans. 2.

CHAPTER VIII.

GENERAL THEORY OF EQUATIONS.

SECTION I.

Composition of Equations.

266. *Any equation of the* n*th degree*, with one unknown quantity, when reduced as in art. 118, *may be represented by the form*

$$A x^n + B x^{n-1} + C x^{n-2} + \&c. + M = 0.$$

If this equation is divided by A, and the coefficients $\frac{B}{A}$, $\frac{C}{A}$, &c., $\frac{M}{A}$ represented by a, b, &c., m, it is reduced to

$$x^n + a x^{n-1} + b x^{n-2} + \&c. + m = 0.$$

267. *Theorem. If any root of the equation*

$$x^n + a x^{n-1} + b x^{n-2} + \&c. + m = 0$$

is denoted by x', *the first member of this equation is a polynomial, divisible by* $x - x'$, *without regard to the value of* x.

Proof. Denote $x - x'$ by $x^{[1]}$, that is,

$$x^{[1]} = x - x',$$

or

$$x = x' + x^{[1]}.$$

If this value of x is substituted in the given equation, if $P x^{[1]}$ is used to denote all the terms multiplied by $x^{[1]}$, or

by any power of $x^{[1]}$, and Q used to denote the remaining terms, the equation becomes

$$P\,x^{[1]} + Q = 0.$$

Now the given equation is, by hypothesis, satisfied by the value of x.

$$x = x',$$

or

$$x^{[1]} = 0;$$

by which the preceding equation is reduced to

$$Q = 0.$$

The terms not multiplied by $x^{[1]}$, or a power of $x^{[1]}$, must, therefore, cancel each other; and the first member of the given equation becomes

$$P\,x^{[1]},$$

which is divisible by $x^{[1]}$, or its equal $x - x'$.

268. *Corollary*. The preceding division is easily effected by subtracting from the polynomial

$$x^n + a\,x^{n-1} + b\,x^{n-2} + \&c. + m,$$

the expression

$$x'^n + a\,x'^{n-1} + b\,x'^{n-2} + \&c. + m = 0,$$

which does not affect its value, but brings it to the form

$$x^n - x'^n + a\,(x^{n-1} - x'^{n-1}) + b\,(x^{n-2} - x'^{n-2}) + \&c.,$$

of which each term is, by art. 49, divisible by $x - x'$. The quotient is, by art. 50,

$$\begin{array}{ll|ll|ll|l} x^{n-1} & + x' & x^{n-2} & + x'^2 & x^{n-3} & + x'^3 & x^{n-4} + \&c. \\ & + a & & + a\,x' & & + a\,x'^2 & \\ & & & + b & & + b\,x' & \\ & & & & & + c & \end{array}$$

269. *Corollary*. The first term of the preceding quotient is x^{n-1}; and if the coefficients of x^{n-2}, x^{n-3}, &c., are denoted by a', b', &c., the quotient is

$$x^{n-1} + a'\,x^{n-2} + b'\,x^{n-3} + \&c.;$$

and the equation of art. 267, is

$$(x-x')(x^{n-1}+a'x^{n-2}+b'x^{n-3}+\&c.)=0;$$

which is satisfied either by the value of x,

$$x=x',$$

or by the roots of the equation

$$x^{n-1}+a'x^{n-2}+b'x^{n-3}+\&c.=0.$$

If now x'' is one of the roots of this last equation, we have in the same way

$$x^{n-1}+a'x^{n-2}+\&c.=(x-x'')(x^{n-2}+a''x^{n-3}+\&c.)=0,$$

and the given equation becomes

$$(x-x')(x-x'')(x^{n-2}+a''x^{n-3}+\&c.)=0;$$

which is satisfied by the value of x'',

$$x=x'';$$

so that x'' is a root of the given equation.

By proceeding in this way to find the roots x''', x^{IV}, &c., the given equation may be reduced to the form

$$(x-x')(x-x'')(x-x''')(x-x^{\text{IV}})\&c=0,$$

in which the number of factors $x-x'$, $x-x''$, &c. is the same with the degree n of the given equation; and, therefore, *the number of roots of an equation is denoted by the degree of the equation;* that is, an equation of the third degree has three roots, one of the fourth degree has four roots, &c. *But all these roots are not necessarily real or rational; they may, on the contrary, be irrational or even imaginary.*

270. *Scholium.* Some of the roots x', x'', x''', &c., are often equal to each other, and in this case the number of unequal roots is less than the degree of the equation.

Thus the number of unequal roots of the equation of the 9th degree,

$$(x-7)(x+4)^3(x-1)^5=0,$$

is but three, namely, 7, -4, and 1, and yet it is to be regarded as having 9 roots, one equal to 7, three equal to -4, and five equal to 1.

271. *Corollary.* The equation

$$x^n=a$$

would appear to have but one root, that is,

$$x=\sqrt[n]{a};$$

but it must, by art. 269, have n roots, or rather, *the* n*th root of* a *must have* n *different values.*

272. EXAMPLES.

1. Find the two roots of the equation

$$x^2=1.$$

Ans. $x=1$, or $=-1$.

2. Find the three roots of the equation

$$x^3=1.$$

Solution. Since one root of this equation is

$$x=1,$$

the equation

$$x^3-1=0$$

must be divisible by $x-1$, and we have

$$x^3-1=(x-1)(x^2+x+1)=0.$$

Now the roots of the equation

$$x^2+x+1=0$$

are

$$x=\tfrac{1}{2}(-1+\sqrt{-3}), \text{ and } =\tfrac{1}{2}(-1-\sqrt{-3}).$$

Hence the required roots are

$$x = 1, = \tfrac{1}{2}(-1+\sqrt{-3}), \text{ and } = \tfrac{1}{2}(-1-\sqrt{-3}).$$

3. Find the four roots of the equation

$$x^4 = 1.$$

Solution. The square root of this equation is

$$x^2 = +1, \text{ or } = -1;$$

so that the required roots are

$$x = 1, = -1, = \sqrt{-1}, \text{ and } = -\sqrt{-1}.$$

4. Find the five roots of the equation

$$x^5 = 1.$$

Solution. Since one root of this equation is

$$x = 1,$$

the equation

$$x^5 - 1 = 0$$

must be divisible by $x - 1$, and we have

$$x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1) = 0.$$

Now the roots of the equation

$$x^4 + x^3 + x^2 + x + 1 = 0$$

can be found by the following peculiar process.

Divide by x^2, and we have

$$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0.$$

If we make

$$y = x + \frac{1}{x},$$

we have

$$y^2 = x^2 + 2 + \frac{1}{x^2},$$

and

$$x^2 + \frac{1}{x^2} = y^2 - 2;$$

which, being substituted in the preceding equation, gives

$$y^2 + y - 1 = 0;$$

the roots of which are

$$y = \tfrac{1}{2}(-1 + \sqrt{5}), \text{ and } = \tfrac{1}{2}(-1 - \sqrt{5}).$$

But the values of x deduced from the equation

$$y = x + \frac{1}{x},$$

or

$$x^2 - y\,x = -1,$$

are

$$x = \tfrac{1}{2}[y + \sqrt{(y^2 - 4)}], \text{ and } = \tfrac{1}{2}[y - \sqrt{(y^2 - 4)}],$$

in which y, being substituted, gives

$$x = \tfrac{1}{4}[-1 - \sqrt{5} \pm \sqrt{(-10 + 2\sqrt{5})}],$$

$$\text{and } = \tfrac{1}{4}[-1 - \sqrt{5} \pm \sqrt{(-10 + 2\sqrt{5})}].$$

5. Find the six roots of the equation

$$x^6 = 1.$$

Ans. $x = 1, = -1, = \tfrac{1}{2}(-1 \pm \sqrt{-3}),$
and $= \tfrac{1}{2}(1 \pm \sqrt{-3}).$

We might proceed in the same way to higher equations, such as the 8th, 9th, 12th, &c.; but, since much more simple solutions are given by the aid of trigonometry, this subject will be postponed to a more advanced part of the course.

273. *Corollary.* Before proceeding farther, we may remark, that the method of solution used in the last example of the preceding article may be applied to any equation of an even degree, in which the successive coefficients of the different powers of x are the same, whether the equation is arranged according to the ascending or according to the descend-

ing powers of x, as is the case in the following equation.

$$A x^{2n} + B x^{2n-1} + C x^{2n-2} + \&c.$$
$$+ C x^2 + B x + A = 0.$$

274. EXAMPLES.

1. Solve the equation

$$A x^4 + B x^3 + C x^2 + B x + A = 0.$$

Solution. Divide by x^2, and we have

$$A \left(x^2 + \frac{1}{x^2}\right) + B \left(x + \frac{1}{x}\right) + C = 0,$$

and, if we make

$$y = x + \frac{1}{x},$$

we have

$$x^2 + \frac{1}{x^2} = y^2 - 2,$$

and

$$A y^2 + B y + C - 2 A = 0;$$

the roots of which are

$$y = \frac{-\frac{1}{2} B \pm \sqrt{(2 A^2 - A C + \frac{1}{4} B^2)}}{A},$$

which are to be substituted in the values of x,

$$x = \tfrac{1}{2} y \pm \sqrt{(\tfrac{1}{4} y^2 - 1)},$$

deduced from the equation

$$y = x + \frac{1}{x}.$$

2. Solve the equation

$$x^6 + 3 x^5 - 7 x^4 + 6 x^3 - 7 x^2 + 3 x + 1 = 0.$$

Solution. Divide by x^3, and we have

$$\left(x^3+\frac{1}{x^3}\right)+3\left(x^2+\frac{1}{x^2}\right)-7\left(x+\frac{1}{x}\right)+6=0;$$

and if we make

$$y=x+\frac{1}{x},$$

we have

$$x^2+\frac{1}{x^2}=y^2-2,$$

$$x^3+\frac{1}{x^3}=y^3-3\left(x+\frac{1}{x}\right)=y^3-3\,y;$$

and the equation becomes, by substitution,

$$y^3+3\,y^2-10\,y=0.$$

The roots of this equation are

$$y=0, =2, \text{ and } =-5;$$

and, therefore, the values of x are

$$x=\pm\sqrt{-1}, =1, \text{ or } =\tfrac{1}{2}(-5\pm\sqrt{21}).$$

3. Solve the equation

$$x^8+2\,x^6-6\,x^4+2\,x^2+1=0.$$

Ans. $x=\pm 1$, or $=\pm\tfrac{1}{2}\sqrt{(-2\pm\sqrt{3})}$.

4. Solve the equation

$$2\,x^4-3\,x^3-x^2-3\,x+2=0.$$

Ans. $x=2$, or $=\tfrac{1}{2}$, or $=\tfrac{1}{2}(-1\pm\sqrt{-3})$.

275. *Corollary.* It follows, from art. 269, that an equation of the second degree has two roots, both of which are given by the process of art. 230; and if the equation is reduced to the form

$$x^2+a\,x+b=0,$$

and the roots denoted by x' and x'', we have

$$x^2+a\,x+b=(x-x')\,(x-x'')=0.$$

Values of Coefficients in Equations.

But the product $(x-x')(x-x'')$ being arranged according to powers of x, is

$$x^2-(x'+x'')x+x'x'';$$

which, being compared with its equal,

$$x^2+ax+b,$$

gives

$$-(x'+x'')=a,$$
$$x'x''=b;$$

that is, the coefficient of x is the negative of the sum of the roots of equation, and the term which does not contain x is the product of the roots.

276. *Corollary.* If the roots of the general equation of the third degree

$$x^3+ax^2+bx+c=0$$

are denoted by

$$x',\ x'',\ x''',$$

we have

$$x^3+ax^2+bx+c=(x-x')(x-x'')(x-x''')=0.$$

But the product

$$(x-x')\,(x-x'')\,(x-x''')$$

is, when arranged according to powers of x,

$$x^3-(x'+x''+x''')x^2+(x'x''+x'x'''+x''x''')x-x'x''x''';$$

whence, by comparison with the given equation, we have

$$a=-(x'+x''+x'''),$$
$$b=x'x''+x'x'''+x''x''',$$
$$c=-x'x''x''';$$

that is, the coefficient of x^2 is the negative of the sum of the roots, the coefficient of x is the sum of the products of the roots multiplied together two and two, and the term which does not contain x is the negative of the continued product of the roots

277. *Corollary.* It may be shown in the same way that, in the equation

$$x^n + a\,x^{n-1} + b\,x^{n-2} + c\,x^{n-3} + \&c. = 0,$$

the coefficient of x^{n-1} is the negative of the sum of the roots; the coefficient of x^{n-2} is the sum of the products of the roots multiplied together two and two; the coefficient of x^{n-3} is the negative of the sum of the products of the roots multiplied together three and three; and so on, the last term being the product of the roots when n *is even, and the negative of this product when* n *is odd.*

SECTION II.

Equal Roots.

278. ***Problem.*** ***To find the equal roots of an equation.***

Solution. Let x' be one of the equal roots which occurs n times as a root of the given equation, the first member of which is therefore divisible by $(x - x')^n$. If the quotient is a function of n denoted by X, the equation is, then,

$$(x - x')^n\, X = 0.$$

The derivative of this first member is, as in art. 177,

$$n\,(x - x')^{n-1}\, X + (x - x')^n\, Y,$$

provided that Y is the derivative of X. The factor $x - x'$ occurs, then, $(n - 1)$ times in this derivative of the first member, that is, once less than in the first member itself. The greatest common divisor of the first member and its derivative must, therefore, consist of the factors $(x - x')$ of

the first member, each being repeated once less than in the first member. No one of them is, then, a factor of the common divisor, unless it is more than once a factor of the first member, that is, unless it corresponds to one of the equal roots.

The equal roots of an equation are, therefore, obtained by finding the greatest common divisor of its first member and its derivative, and solving the equation obtained from putting this common divisor equal to zero.

279. *Corollary.* The common divisor must, itself, have equal roots, whenever a root is more than twice a root of the given equation.

280. EXAMPLES.

1. Find all the roots of the equation

$$x^3 - 7x^2 + 16x - 12 = 0$$

which has equal roots.

Solution. The derivative of this equation is

$$3x^2 - 14x + 16,$$

the greatest common divisor of which and the given first member is

$$x - 2.$$

The equation

$$x - 2 = 0,$$

gives

$$x = 2.$$

Now since the given equation has two roots equal to 2, it must be divisible by

$$(x - 2)^2 = x^2 - 4x + 4,$$

and we have

$$x^3 - 7x^2 + 16x - 12 = (x-2)^2(x-3) = 0;$$

whence

$$x = 3$$

is the other root of the given equation.

2. Find all the roots of the equation

$$x^7 - 9x^5 + 6x^4 + 15x^3 - 12x^2 - 7x + 6 = 0$$

which has equal roots.

Solution. The derivative of this equation is

$$7x^6 - 45x^4 + 24x^3 + 45x^2 - 24x - 7,$$

the greatest common divisor of which and the given equation gives

$$x^3 - x^2 - x + 1 = 0,$$

which is an equation of the third degree, and we may consider it as a new equation, the equal roots of which are to be found, if it has any.

Now its derivative is

$$3x^2 - 2x - 1,$$

and the common divisor of this derivative and the first member gives

$$x - 1 = 0, \text{ or } x = 1.$$

Hence the first member of

$$x^3 - x^2 - x + 1 = 0$$

must be divisible by

$$(x-1)^2,$$

and we have indeed

$$x^3 - x^2 - x + 1 = (x-1)^2(x+1) = 0.$$

The equal roots of the given equation are, therefore,

$$x = 1, \text{ and } = -1;$$

and its first member is divisible by

$$(x-1)^3(x+1)^2,$$

Examples of finding Equal Roots.

and is found by division to be

$$(x-1)^3\,(x+1)^2\,(x^2+x-6).$$

The remaining roots are, therefore, found from solving the quadratic equation

$$x^2+x-6=0,$$

which gives

$$x=2, \text{ or } =-3.$$

3. Find all the roots of the equation

$$x^3+3\,x^2-9\,x-27=0$$

which has equal roots.

Ans. $x=3$, or $=-3$.

4. Find all the roots of the equation

$$x^3-15\,x^2+75\,x-125=0$$

which has equal roots.

Ans. $x=5$.

5. Find all the roots of the equation

$$x^4-9\,x^3+29\,x^2-39\,x+18=0$$

which has equal roots.

Ans. $x=1$, or $=2$, or $=3$

6. Find all the roots of the equation

$$x^4-2\,x^3-59\,x^2+60\,x+900=0$$

which has equal roots.

Ans. $x=6$, or $=-5$.

7. Find all the roots of the equation

$$x^4-6\,x^2-8\,x-3=0$$

which has equal roots.

Ans. $x=3$, or $=-1$.

8. Find all the roots of the equation

$$x^4+12\,x^3+54\,x^2+108\,x+81=0$$

which has equal roots.

Ans. $x=-3$.

9. Find all the roots of the equation

$$x^5 - 2x^4 - 2x^3 + 4x^2 + x - 2 = 0$$

which has equal roots.

Ans. $x = \pm 1$, or $= 2$.

10. Find all the roots of the equation

$$x^6 - 6x^4 + 4x^3 + 9x^2 - 12x + 4 = 0$$

which has equal roots.

Ans. $x = 1$, or $= -2$.

11. Find the equal roots of the equation

$$x^8 - 8x^7 + 26x^6 - 45x^5 + 45x^4 - 21x^3 - 10x^2 + 20x - 8 = 0.$$

Ans. $x = 1$, or $= 2$.

SECTION III.

Real Roots.

281. *Theorem. When an equation is reduced, as in art.* 266, *and the values of its first member, obtained by the substitution of two different numbers for its unknown quantity, are affected by contrary signs, the given equation must have a real root comprehended between these two numbers.*

Proof. For, if the value of the less of the two numbers, which are substituted for the unknown quantity is supposed to be increased by imperceptible degrees until it attains the value of the greater number, the value of the first member must likewise change by imperceptible degrees, and must pass through all the intermediate values between its two extreme values. But the extreme values are affected with opposite signs, so that zero must be contained between them, and must be one of the values attained by the first member;

that is, there must be a number which, substituted in the first member, reduces it to zero, and this number is consequently a root of the given equation.

282. *Corollary. If the given equation has no real root, the value of its first member will always be affected by the same sign, whatever numbers be substituted for its unknown quantity.*

283. *Theorem. When an uneven number of the real roots of an equation are comprehended between two numbers, the values of its first member obtained, by substituting these numbers for* x, *must be affected with contrary signs; but if an even number of roots is contained between them, the values obtained from this substitution must be affected with the same sign.*

Proof. Denote by x', x'', x''' &c. all the roots of the given equation which are contained between the given numbers p and q; the first member of the given equation must, by art. 269, be divisible by

$$(x - x')\,(x - x'')\,(x - x''')\ \&c.$$

If we denote the quotient of this division by Y, the equation

$$Y = 0$$

gives all the remaining roots of the given equation, so that

$$Y = 0$$

cannot have any real root contained between p and q.

The given first member being, therefore, represented by

$$(x - x')\,(x - x'')\,(x - x''')\ \ldots . \times Y$$

becomes

$$(p - x')\,(p - x'')\,(p - x''')\ \ldots . \times Y',$$

when we substitute p for x, and denote the corresponding

value of Y by Y'; and when we substitute q for x, and denote the corresponding value of Y by Y'', it becomes

$$(q - x')(q - x'')(q - x''') \ldots \times Y''.$$

The quotient of these two results is

$$\frac{(p - x')(p - x'')(p - x''') \ldots Y'}{(q - x')(q - x'')(q - x''') \ldots Y''},$$

which can be written

$$\frac{p - x'}{q - x'} \times \frac{p - x''}{q - x''} \times \frac{p - x'''}{q - x'''} \times \ldots \frac{Y'}{Y''}.$$

Now since each of the roots x', x'', x''', &c., is included between p and q, the numerator and denominator of each of the fractions

$$\frac{p - x'}{q - x'},\ \frac{p - x'}{q - x''},\ \frac{p - x'''}{q - x'''},\ \&c.,$$

must be affected with contrary signs, and therefore each of these fractions must be negative.

But since Y' and Y'' must, by art. 282, have the same sign, the fraction

$$\frac{Y'}{Y''}$$

is positive.

The product of all these fractions is therefore *positive*, when the number of the fractions

$$\frac{p - x'}{q - x'},\ \frac{p - x''}{q - x''},\ \&c.,$$

is *even*, that is, when *the number of the roots*, x', x'', x''', &c., is *even*; and this product is *negative*, when the number of these roots is *uneven*. The values which the given first member obtains by the substitution of p and q for x must, consequently, be affected with *contrary* signs in the latter case; and with the *same* sign in the former case.

Number of Real Roots of an Equation of an Odd Degree.

284. *Theorem.* ***Every equation of an uneven degree, has at least one real root affected with a sign contrary to that of its last term, and the number of all its roots is uneven.***

Proof. Let the equation be

$$x^n + a x^{n-1} + \&c. \ldots + m = 0,$$

in which n is uneven.

First, to prove that there is a real root, and that the number of real roots is uneven. Every real root must be contained between $+\infty$ and $-\infty$. Now the substitution of

$$x = \infty,$$

gives the value of the first member

$$\infty^n + a \infty^{n-1} + b \infty^{n-2} + \&c. \ldots + m;$$

the first term of which is infinitely greater than any other term, or than the sum of all the other terms. The sign of this result is therefore the same as that of its first term, or *positive.*

Again, the substitution of

$$x = -\infty$$

gives, since n is uneven,

$$-\infty^n + a \infty^{n-1} - b \infty^{n-2} + \&c. \ldots + m,$$

which may be shown by the above reasoning to be *negative.*

The given equation must then, by art. 281, have at least one real root, and by art. 283, the number of its real roots must be uneven.

Secondly. To prove that one, at least, of the real roots is affected with a contrary sign to that of the last term. The substitution of

$$x = 0,$$

reduces the given first member to its last term m.

Comparing this with the above results, we see that, if m is *positive*, the given equation must, by art. 281, have a real root contained between 0 and $-\infty$, that is, a *negative* root; but if m is *negative*, there must be a real root contained between 0 and $+\infty$, that is, a *positive* root; so that there must always be a root affected with a sign contrary to that of m.

285. *Theorem. The number of real roots if there are any, of an equation of an even degree must be even, and if the last term is negative, there must be at least two real roots, one positive and the other negative.*

Proof. Let the equation be

$$x^n + a\,x^{n-1} + b\,x^{n-2} + \&c. \ldots + m = 0,$$

in which n is even.

First. To prove that the number of real roots is even. The substitution of

$$x = \infty$$

gives for the value of the first member

$$\infty^n + a\,\infty^{n-1} + b\,\infty^{n-2} + \&c. \ldots + m,$$

which is *positive*.

The substitution of

$$x = -\infty$$

gives for the value of the first member

$$\infty^n - a\,\infty^{n-1} + b\,\infty^{n-2} + \&c. \ldots + m,$$

which is also *positive*.

Hence, if the given equation has any real root, there must, by art. 283, be an even number of them.

Secondly. To prove that when m is negative, there must be two real roots, the one positive, the other negative. The substitution of

$$x = 0$$

reduces the given first member to its last term m, and this result is therefore *negative* in the present case.

Comparing this with the above results, we see that there must be a real root between 0 and $+\infty$, and also one between 0 and $-\infty$; that is, the given equation has two real roots, the one positive and the other negative.

286. *Corollary.* Since the number of real roots of an equation of an uneven degree is uneven, and that of an equation of an even degree is even, *the number of imaginary roots of every equation, which has imaginary roots, must be even.*

287. *Theorem. The number of real positive roots of an equation is even, when its last term is positive; and it is uneven, when the last term is negative.*

Proof. The substitution of

$$x = \infty$$

gives, for the first member of the given equation, a *positive* result; while the substitution of

$$x = 0$$

reduces the first member to its last term.

Hence if this last term is *positive*, the number of real roots contained between 0 and ∞, that is, of positive roots, must, by art. 283, be even; and if this last term is negative, the number of these roots must be uneven.

288. *Theorem. If a function vanishes, that is, is equal to zero for a given value* x' *of its variable* x*; the function and its derivative must have like signs for a value of the variable which exceeds* x' *by*

an infinitely small quantity, and unlike signs for a value of the variable which is less than x' *by an infinitely small quantity.*

Proof. Let the given function be u, and its derivate U, and, as in art. 176, when the variable is increased by the infinitesimal i, the function becomes

$$u + U i.$$

This value of the function, when

$$u = 0$$

is reduced to $U i$, which has, obviously, the same sign with U.

In the same way when the variable is decreased by i, the function becomes

$$u - U i,$$

which, when

$$u = 0,$$

is reduced to $-U i$, having the opposite sign to U.

289. *Definition.* A pair of two successive signs in a row of signs, is called a *permanence* when the two signs are alike, and a *variation* when they are unlike.

290. *Sturm's Theorem.* Denote the first member of the equation

$$x^n + a\, x^{n-1} + \&c. = 0$$

by u and its derivative by U. Find the greatest common divisor of u and U, and, in performing this process, let the several remainders which are of continually decreasing dimensions in regard to x, be denoted, after reversing their signs, by

$$U', \ U'', \ U''', \ \&c..$$

Find the row of signs corresponding to the values of

$$u,\ U,\ U',\ U'',\ \&c.,$$

for any value p of the variable, and also for a value q of the variable.

The difference between the number of permanences of the first row of signs, and that of the second, is exactly equal to the number of real roots of the given equation comprised between p *and* q.

Proof. The method in which U', U'', &c., are obtained gives, at once, by denoting the successive quotients in the process by m, m', &c.

$$\begin{aligned} u &= m\ \ U - U' \\ U &= m'\ U' - U'' \\ U' &= m''\ U'' - U''' \\ &\&c. \end{aligned}$$

First. Two successive terms of the series cannot vanish at the same time, except for a value of x which is one of the equal roots of the given equation. For when U'' and U''', for instance, are zero, the equation

$$U' = m\ U'' - U'''$$

gives

$$U' = 0;$$

and, in the same way, it is shown that

$$U = 0 \text{ and } u = 0,$$

so that the function and the derivative are both zero at the same time, which, by art. 278, corresponds to the case of one of the equal roots of the equation.

Secondly. If any term of the series, except the first or last, has a different sign in the row corresponding to the value p of the variable from that which it has in the row

corresponding to the value q of the variable, it must, by art. 281, vanish for some value of the variable contained between p and q. But for this value of the variable, the preceding term must have a different sign from the succeeding term; thus, when

$$U'' = 0$$

the equation

$$U' = m'' U'' - U'''$$

gives

$$U' = - U'''.$$

By the change of sign which the term undergoes in vanishing, therefore, it can only change from forming a permanence with one of its adjacent terms to forming one with the other of these terms, and *the change of sign of a term, which is neither the first nor the last of the series, does not increase or diminish the number of permanences of the row of signs.*

Thirdly. When the first term u of the series, in changing its sign, vanishes, while the second term U does not vanish, the corresponding value of the variable is, by art. 278, a root of the equation which is not one of the equal roots. If, moreover, the variable is decreasing in value, the signs of these two terms constitute a permanence before the change and a variation after the change. *When the variable, therefore, in decreasing passes through a value which is one of the unequal roots of the equation, the number of permanences in the row of signs is increased by unity.*

Fourthly. When the given equation has no equal roots, u and U can, by art. 278, have no common divisor, and therefore the last term of the series will not contain the variable; it must, therefore, be of a constant value and no change of sign can arise from it. *In this case, then, the number of permanences must by the preceding division of the proof be greater in the row which corresponds to the greater*

of the two limits p *and* q, *than in the row which corresponds to the less of these two limits, by a number which is exactly equal to the number of real roots contained between* p *and* q.

Fifthly. When the given equation has equal roots, u and U must, by art. 278, have a common divisor which will be the last term of the series. This divisor must also, by art. 59, be a divisor of all the other terms of the series; and if the series is divided by it a new series

$$v,\ V,\ V',\ V'',\ \&c.,$$

is obtained, which has in all cases either the same signs as the given series or the reverse signs, so that each pair of successive signs is of the same name, whether permanence or variation, in each series. And by dividing the equations before found by this same common divisor, they become

$$v = m\ V - V'$$
$$V = m'\ V' - V''$$
$$\&c.$$

The first term of the new series has, by art. 278, the same roots with the given series except that it has no equal roots, and the last term is unity. The reasoning of the preceding portion of this article may, therefore, be applied to the new series; and it follows that *the theorem is applicable to the case of an equation which has equal roots, as well as to one which has unequal roots.*

291. *Corollary. If infinity is substituted for* p *and negative infinity for* q *in the series of divisors, the resulting rows of signs show at once the whole number of real roots of the given equation.*

292. EXAMPLES.

1. Find the number of real roots of the equation

$$2x^4 - 20x + 19 = 0$$

and also the number contained between 1 and 2.

Solution. In this case, we have

$$\begin{aligned} u &= 2x^4 - 20x + 19, \\ U &= 8x^3 - 20; \end{aligned}$$

and by the method of the common divisor

$$\begin{array}{r|r|l}
2x^4 - 20x + 19 & 2x^3 - 5 \quad\quad & x \\
2x^4 - 5x & & \\
\hline
-15x + 19 & 30x^3 - 75 \quad\quad & -2x^2. \\
& +30x^3 - 38x^2 & \\
\cline{2-2}
& +38x^2 - 75 \quad & \\
& +570x^2 - 1125 & -38x \\
& 570x^2 - 722x & \\
\cline{2-2}
& 722x - 1125 & \\
& 10830x - 16875 & -722 \\
& 10830x - 13718 & \\
\cline{2-2}
& -3157 &
\end{array}$$

$$\begin{aligned} U' &= 15x - 19 \\ U'' &= 3157. \end{aligned}$$

When, therefore, $x = \infty$ the row of signs is

$$+, +, +, +;$$

and when $x = -\infty$, it is

$$+, -, -, +;$$

there are then two real roots.

Again when $x = 2$, the row of signs is

$$+, +, +, +;$$

and when $x = 1$, it is

$$+, -, -, +;$$

the two real roots are therefore both between 1 and 2.

2. Find the number of real roots of the equation

$$x^3 + a\,x + b = 0.$$

Solution. In this case, we have

$$\begin{aligned} u &= x^3 + a\,x + b \\ U &= 3\,x^2 + a \\ U' &= -2\,a\,x - 3\,b \\ U'' &= -4\,a^3 - 27\,b^2. \end{aligned}$$

First case. When a is such that U'' is negative, that is when

$$-4\,a^3 < 27\,b^2, \text{ or } -\tfrac{1}{27}\,a^3 < \tfrac{1}{4}\,b^2, \text{ or } (-\tfrac{1}{3}\,a)^3 < (\tfrac{1}{2}\,b)^2$$

the row of signs when $x = \infty$ is

$$+, +, \mp \text{ (the reverse of } a), -;$$

and when $x = -\infty$ it is

$$-, +, \pm \text{ (like } a), -,$$

so that there is only one real root.

The row of signs when $x = 0$ is, when b is positive,

$$+, \pm \text{ (like } a), -, -,$$

so that, in this case, the real root is negative.

This row, when b is negative, is

$$-, \pm \text{ (like } a), +, -,$$

so that, in this case, the real root is positive, which agrees with art. 284.

Second case. When a is negative and of such a value that

$$(\tfrac{1}{2}\,b)^2 = -(\tfrac{1}{3}\,a)^3,$$

that is

$$U'' = 0,$$

in which case the equation has the equal root, obtained from the equation

$$U' = -2\,a\,x - 3\,b = 0$$

or

$$x = -\frac{3\,b}{-2\,a};$$

and in this case the row of signs when $x = \infty$ is

$$+, +, +;$$

and that when $x = -\infty$ is

$$-, +, -;$$

so that the two different roots of the equation are, in this case, real.

The row of signs when $x = 0$ is

$$\pm \text{ (like } b), -, \mp \text{ (unlike } b);$$

so that one of the roots is positive and the other negative.

Third case. When a is negative and of such a value that U'' is positive or

$$(-\tfrac{1}{3} a)^3 > (\tfrac{1}{2} b)^2$$

in which case, the row of signs when $x = \infty$ is

$$+, +, +, +;$$

and when $x = -\infty$ it is

$$-, +, -, +;$$

so that all three of the roots of the equation are real.

The row of signs when $x = 0$ is

$$\pm \text{ (like } b), -, \mp \text{ (like } b), -.$$

If, then, b is positive the equation has one positive real root and two negative ones; and if b is negative, it has two positive real roots and one negative one.

3. Find the number of real roots of the equation

$$x^n + a = 0.$$

Solution. In this case, we have

$$\begin{aligned} u &= x^n + a \\ U &= n\, x^{n-1} \\ U' &= -a. \end{aligned}$$

First case. If n is even, the row of signs, when $x = \infty$ is, then,

$$+, +, \mp \text{ (unlike } a);$$

when $x = -\infty$ it is

$$+, -, \mp \text{ (unlike } a\text{)};$$

so that there is no real root when a is positive, and two real roots when a is negative, which agrees with art. 285.

Second case. If n is odd, the row of signs when $x = \infty$ is

$$+, +, \mp \text{ (unlike } a\text{)};$$

when $x = -\infty$ it is

$$-, +, \mp \text{ (unlike } a\text{)};$$

so that, in either case, there is only one real root, which is, by art. 284, of a sign unlike that of a.

4. Find the number of real roots of the equation

$$x^n + a\,x + b = 0.$$

Solution. In this case, we have

$$\begin{aligned} u &= x^n + a\,x + b, \\ U &= n\,x^{n-1} + a, \\ U' &= -(n-1)\,a\,x - n\,b, \\ U'' &= -a^n\,(n-1)^{n-1} - n^n\,(-b)^{n-1}. \end{aligned}$$

First case. When n is even and greater than 2, and U'' positive, that is, when b is positive, and

$$\left(\frac{a}{n}\right)^n < \left(\frac{b}{n-1}\right)^{n-1};$$

the row of signs when $x = \infty$ is

$$+, +, \mp \text{ (unlike } a\text{)}, +;$$

when $x = -\infty$ it is

$$+, -, \pm \text{ (like } a\text{)}, +;$$

so that when a is positive, there is no real root, and when a is negative there are two real roots. In the latter case, the row of signs when $x = 0$ is

$$+, -, -, +;$$

so that both the real roots are positive

Second case. When n is even and greater than 2, and U'' zero, that is when b is positive and

$$\left(\frac{a}{n}\right)^n = \left(\frac{b}{n-1}\right)^{n-1};$$

in which case, there is the equal root

$$x = -\frac{nb}{(n-1)a} = -\sqrt[n-1]{\frac{a}{n}}.$$

The row of signs when $x = \infty$ is

$$+,\ +,\ \mp \text{ (unlike } a);$$

when $x = -\infty$ it is

$$+,\ -,\ \pm \text{ (like } a);$$

so that in either case there is no other real root than the above equal root.

Third case. When n is even and greater than 2, and U'' negative, that is,

$$\left(\frac{a}{n}\right)^n > \left(\frac{b}{n-1}\right)^{n-1},$$

the row of signs when $x = \infty$ is

$$+,\ +,\ \mp \text{ (unlike } a),\ -;$$

when $x = -0$ it is

$$+,\ -,\ \pm \text{ (like } a),\ -;$$

so that when a is negative there is no real root, and when a is positive there are two real roots. In the latter case, the row of signs when $x = 0$ is

$$\pm \text{ (like } b),\ +,\ \mp \text{ (unlike } b),\ -;$$

so that when b is positive, both the roots are negative, and when b is negative, one of the roots is positive and the other negative, which agrees with art. 287.

Fourth case. When n is odd, and U'' positive, that is, when a is negative and

$$\left(-\frac{a}{n}\right)^n > \left(\frac{b}{n-1}\right)^{n-1};$$

Number of Real Roots.

in which case, the row of signs when $x = \infty$ is

$$+, +, +, +;$$

when $x = -\infty$ it is

$$-, +, -, +;$$

so that the equation has three real roots; the row of signs when $x = 0$ is

$$\pm \text{ (like } b), -, \mp \text{ (unlike } b), +;$$

so that when b is negative, one of the real roots is positive and the other two negative; and when b is positive, one of the real roots is negative and the other two positive.

Fifth case. When n is odd, and U'' zero, that is, when a is negative and

$$\left(-\frac{a}{n}\right)^n = \left(\frac{b}{n-1}\right)^{n-1},$$

in which case, there is the equal root

$$x = \frac{n\,b}{-(n-1)\,a} = \sqrt[n-1]{-\frac{a}{n}},$$

the row of signs when $x = \infty$ is

$$+, +, +;$$

when $x = -\infty$ it is

$$-, +, -;$$

so that there is another real root besides the above equal root. The row of signs when $x = 0$ is

$$\pm \text{ (like } b), -, \mp \text{ (unlike } b);$$

so that one of the roots is positive and the other negative.

Sixth case. When n is odd, and U'' negative, that is,

$$\left(-\frac{a}{n}\right)^n < \left(\frac{b}{n-1}\right)^{n-1},$$

in which case, the row of signs when $x = \infty$ is

$$+, +, \mp \text{ (unlike } a), - ;$$

when $x = -\infty$ it is

$$-, +, \pm \text{ (like } a), -,$$

so that there is only one real root, which, by art. 284, has a sign contrary to that of its last term.

4. Find the number of real roots of the equation

$$x^3 - 6x^2 + 19x - 44 = 0.$$

Ans. It has one positive real root.

5. Find the number of real roots of the equation

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$$

Ans. It has four positive real roots.

6. Find the number of real roots of the equation

$$x^4 + x^3 - 24x^2 + 43x - 21 = 0.$$

Ans. It has three positive real roots and one negative one.

7. Find the number of real roots of the equation

$$x^4 + 8x^2 + 16x - 440 = 0.$$

Ans. One positive root and one negative root.

293. Sturm's theorem is perfect in always giving the number of real roots, but often requires so much labor, that theorems, which are much less perfect, may be used with great advantage.

294. *Stern's Theorem.* Denote the first member of the equation

$$x^n + a x^{n-1} + \&c. = 0$$

by u and its first, second, &c. derivatives by U, U', &c.

Find the row of signs corresponding to the values of

$$u,\ U,\ U',\ U'',\ \&c.,$$

for any value p of the variable, and also for a value q of the variable.

The number of real roots of the equation, comprised between p *and* q, *cannot be greater than the difference between the number of permanences of the first row of signs and that of the second row.*

Proof. First. It may be shown as in the third division of the proof of art. 290, that *one permanence at least is always lost from the row of signs when the variable in decreasing passes through a value which is one of the roots of the equation.*

Secondly. When any term of the series except the first or the last, vanishes, it passes, by art. 288, with the decreasing variable, from having the same sign with its derivative, which is the next term of the series, to having the reverse sign of it. Even then, if it had before the change the reverse sign of the preceding term and after the change the same sign, it introduces a permanence which is only sufficient to take the place of the other permanence which is lost. *The number of permanences of the row of signs is not, therefore, augmented by the vanishing of such an intermediate term.*

Thirdly. The last term of the series must be constant, for the number of dimensions is diminished by each derivation; and, therefore, as x decreases from a value p to a smaller value q, the number of permanences of the row of signs must be diminished by as large a number at least as the number of roots comprised between p and q.

295. *Definition.* An equation

$$x^n + a\,x^{n-1} + \&c. + k\,x^2 + k\,x + l = 0.$$

is said to be *complete in its form*, when it contains terms multiplied by every different power of x from the highest to unity, and also a constant term, such as l.

296. *Descartes' Theorem. A complete equation cannot have a greater number of positive roots than there are variations in the row of signs of its terms, nor a greater number of positive roots than there are permanences in this row of signs.*

Proof. If the equation is that of art. 295, the values of u, U, U', &c. in art. 294, are

$$\begin{aligned} u &= x^n + a\,x^{n-1} + \&c. + h\,x^2 + k\,x + l \\ U &= n\,x^{n-1} + (n-1)\,a\,x^{n-2} + \&c. + 2\,h\,x + k \\ U' &= n\,(n-1)\,x^{n-2} + (n-1)(n-2)ax^{n-3} + \&c. + 2\,h \\ &\&c. \end{aligned}$$

The row of signs when $x = \infty$ is

$$+,\ +,\ +,\ +,\ +,\ \&c.,$$

consisting wholly of permanences.

When $x = -\infty$ it is

$$\pm,\ \mp,\ \pm,\ \mp,\ \&c.,$$

in which the upper row of signs is used when n is even, and the lower row when n is odd. In either case, this row consists wholly of variations.

The row of signs when $x = 0$ is

$$\pm \text{ (like } l),\ \pm \text{ (like } k),\ \pm \text{ (like } 2\,h)$$

that is, it is the same as the row of signs formed by the terms of the equation taken in the inverse order.

The limit of the number of positive roots is, therefore, by art. 294, equal to the excess of the whole number of pairs of successive signs of the terms, over the number of permanences; that is, it is equal to the number of variations; and in the same way, the number of negative roots cannot exceed the number of permanences.

297. *Corollary.* The whole number of successions of signs of an equation, that is, the sum of the permanences and variations, is one less than the number of terms, or the same as the degree of the equation, that is, the same as the number of roots.

If, therefore, all the roots are real, the number of positive roots must be the same as the number of variations, and the number of negative roots must be the same as the number of permanences.

298. *Scholium.* Whenever a term is wanting in an equation, its place may be supplied by zero, and either sign may be prefixed.

299. *Corollary.* When the substitution of $+0$ for a term which is wanting gives a different number of permanences from that which is obtained by the substitution of -0, and consequently a different number of variations also, the equation must have imaginary roots.

300. *Theorem. When the sign of the term which precedes a deficient term is the same with that which follows it, the equation must have imaginary roots.*

Proof. For if the terms which precede and follow the deficient term are both *positive*, the substitution of $+0$ gives two permanences; while the substitution of -0 gives

two variations. The reverse is the case when both these terms are negative. The equation must, therefore, in either case, have imaginary roots.

301. *Theorem. When two or more successive terms of an equation are wanting, the equation must have imaginary roots.*

Proof. For the second deficient term may be supplied with zero affected by the same sign as that of the term preceding the deficient terms; and the first deficient term is then preceded and followed by terms having the same sign, so that there must, by the preceding article, be imaginary roots.

302. *Theorem. When an uneven number* (m) *of successive terms is wanting in an equation, the number of imaginary roots must be at least as great as* $(m+1)$, *if the term preceding the deficient terms has the same sign with the term following them; and the number of imaginary roots must be at least as great as* $(m-1)$, *if the term preceding the deficient terms has the reverse sign of the term following them.*

Proof. First. If the sign of the term preceding the deficient terms is the same with the sign of the term following them; supply the place of each deficient term with zero affected by this same sign. All the $(m+1)$ successions, dependent upon the deficient terms, must in this case be permanences. But if the sign of every other zero beginning with the first is reversed, namely, of the first, third, fifth, &c., all these permanences are changed into variations; so that $(m+1)$ roots can be neither positive nor negative, and are, consequently, imaginary

Secondly. If the sign of the term preceding the deficient terms is the reverse of the sign of the term following them; supply the place of the two last deficient terms with zero affected by the same sign as that of the term preceding the deficient terms. This case becomes the same as the preceding with $(m-2)$ deficient terms, and there must therefore be $(m-1)$ imaginary roots.

303. *Theorem. When an even number of successive terms is wanting in an equation, the number of imaginary roots must be at least as great as the number of these deficient terms.*

Proof. Let the place of the first deficient term be supplied by zero affected with the same sign as that of the term which follows the deficient terms.

The number of deficient terms is thus reduced to the uneven number $m-1$; and, as the term preceding the deficient terms is now of the same sign with that of the term following them, the number of imaginary roots of the equation must, by the preceding article, be at least as great as

$$(m-1)+1=m.$$

304. A number, which is greater than the greatest of the positive roots of an equation, is called *a superior limit of the positive roots;* and one, which is less than the least of the positive roots, is called *an inferior limit of the positive roots.*

In the same way, *a superior limit of the negative roots* is a number which, neglecting the signs, is greater than the greatest negative root; and *an inferior limit of the negative roots* is a number which is less than the least negative root.

305. ***Problem.*** *To find a superior limit of the positive roots.*

Solution. The sum of all the negative terms being equal to the sum of all the positive terms, must exceed each positive term. Let, then, $-S$ be the greatest negative coefficient of the equation of the nth degree, and m the exponent of the highest negative term; the sum of the negative terms, neglecting their signs, must evidently be less than that of the series

$$S + Sx + Sx^2 + \&c. \ldots + Sx^m,$$

for each term of this series is greater than the corresponding negative term of the equation.

But this series is a geometrical progression of which S is the first term, Sx^m the last term, and x the ratio; so that its sum is, by example 3, of art. 261,

$$\frac{Sx^{m+1} - S}{x - 1};$$

and must be greater than any positive term, as x^n, or

$$x^n < \frac{Sx^{m+1} - S}{x - 1} < \frac{Sx^{m+1}}{x - 1}.$$

Hence

$$(x - 1)\, x^n < Sx^{m+1},$$

or

$$(x - 1)\, x^{n-m-1} < S.$$

But, since

$$x - 1 < x \text{ and } (x - 1)^{n-m-1} < x^{n-m-1},$$

we must have

$$(x - 1)^{n-m} < (x - 1)\, x^{n-m-1} < S;$$

and, therefore,

$$x - 1 < \sqrt[n-m]{S},$$

or

$$x < 1 + \sqrt[n-m]{S}.$$

If we, then, denote by L this superior limit of the positive roots, we have

$$L = 1 + \sqrt[n-m]{S};$$

that is, *a superior limit of the positive roots is unity, increased by that root of the greatest negative coefficient, whose index is equal to the excess of the degree of the equation above the exponent of the first negative term.*

306. *Problem. To find an inferior limit of the positive roots.*

Solution. Substitute in the given equation for x, the value

$$x = \frac{1}{y},$$

and find, by the preceding article, a superior limit of the positive values of y, after the equation is reduced to the usual form; and denote this limit by L'.

We have, then,

$$y < L',$$

and, therefore,

$$\frac{1}{y} > \frac{1}{L'},$$

or

$$x > \frac{1}{L'},$$

so that $\frac{1}{L'}$ is an inferior limit of the positive roots of the given equation.

307. *Problem. To find the limits of the negative roots of an equation.*

Solution. Substitute for x

$$x = -y,$$

and the positive roots of the equation thus formed are the negative roots of the given equation; and, therefore, the limits of its positive roots become, by changing their signs, the required limits.

308. *Corollary*. By the substitution of different numbers for p and q, in arts. 290 or 294, the limits between which each root is obtained can be narrowed to any extent which may be desired, until they may be adopted as the first approximations to the roots in the method of art. 179. Thus, it is easy to obtain the first left hand significant figure.

309. EXAMPLES.

1. Find the left hand significant figure of the real roots of the equation

$$5x^3 - 6x + 2 = 0.$$

Solution. *First*. In this case, 6 is the greatest negative coefficient and $-6x$ is the first negative term, so that, by art. 305,

$$1 + \sqrt{6} = 3{\cdot}5$$

is a superior limit of the positive roots.

To find the limit of the negative roots, let

$$x = -y,$$

and the equation becomes, by reversing its signs,

$$5y^3 - 6y - 2 = 0;$$

so that

$$-(1 + \sqrt{6}) = -3{\cdot}5$$

is the superior limit of the negative roots, and the roots are all contained between 4 and -4.

Secondly. Sturm's theorem gives

$$\begin{aligned} u &= 5x^3 - 6x + 2 \\ U &= 15x^2 - 6 \\ U' &= 2x - 1 \\ U'' &= 3\,; \end{aligned}$$

so that the row of signs when $x = 4$ is

$$+, +, +, +\,;$$

when $x = -4$ it is

$$-, +, -, +\,;$$

so that the equation has three real roots.

The row of signs when $x = 0$ is

$$+, -, -, +\,;$$

so that two of the roots are positive and one is negative.

The substitution of positive integers, gives for the rows of signs when $x = 1$

$$+, +, +, +\,;$$

so that both the positive roots are contained between 0 and 1.

The substitution of the positive decimals 0·1, 0·2, 0·3, &c., gives the following rows of signs.

$x = 0{\cdot}1$	$+, -, -, +$
$x = 0{\cdot}2$	$+, -, -, +$
$x = 0{\cdot}3$	$+, -, -, +$
$x = 0{\cdot}4$	$-, -, -, +$
$x = 0{\cdot}5$	$-, -, \pm, +$
$x = 0{\cdot}6$	$-, -, +, +$
$x = 0{\cdot}7$	$-, +, +, +$
$x = 0{\cdot}8$	$-, +, +, +$
$x = 0{\cdot}9$	$+, +, +, +\,;$

so that one real root is contained between 0·3 and 0·4, and the other between 0·8 and 0·9; their first approximate values are, then, 0·3 and 0·8.

The substitution of the negative integers gives, in the

same way -1, for an approximate value of the negative root

2. Find the left hand significant figures of the roots of the equation

$$x^4+8x^2+16x-440=0.$$

Ans. 3 and -4.

3. Find the first approximation to the roots of the equation

$$x^5-15x^3+132x^2+36x+396=0.$$

Ans. $1, -1, -5$.

4. Find, by Stern's theorem, the greatest possible number of real roots which the equation

$$x^{10}-10x^8-x^4+x-11=0$$

can have between $+1$ and -1.

Solution. In this case we have, by art. 294,

$$\begin{aligned}
u &= x^{10}-10x^8-x^4+x-11\\
U &= 10x^9-80x^7-4x^3+1\\
U' &= 90x^8-560x^6-12x^2\\
U'' &= 720x^7-3360x^5-24x\\
U''' &= 5040x^6-16800x^4-24\\
U^{\text{iv}} &= 30240x^5-67200x^3\\
U^{\text{v}} &= 151200x^4-201600x^2\\
U^{\text{vi}} &= 604800x^3-403200x\\
U^{\text{vii}} &= 1814600x^2-403200\\
U^{\text{viii}} &= 362800x\\
U^{\text{ix}} &= 3628800;
\end{aligned}$$

the row of signs when $x=1$ is

$$-,-,-,-,-,-,-,+,+,+,+;$$

when $x=-1$ it is

$$-,+,-,+,-,+,-,-,+,-,+;$$

so that the number of these roots cannot exceed 8.

Integral Root.

Again, when x is infinitely little greater than zero, for which value some of the differential coefficients vanish, the row of signs is

$$-, +, -, -, -, -, -, -, +, -;$$

so that there cannot be more than three roots between 0 and 1; and since the sign of the first term is the same when $x = 0$, that it is when $x = 1$, there cannot, by art. 283, be an odd number of real roots between 0 and 1, and consequently there cannot be more than 2. The row of signs when x is less than zero by an infinitesimal is

$$-, +, -, +, -, + -, +, -, -, +;$$

so that there can be no real root between 0 and -1.

5. Find, by Stern's and Descartes' theorems, the greatest possible number of real roots of the equation

$$x^6 - 5x^4 + x^3 - x^2 - 1 = 0.$$

comprised between 0 and 1.

Ans. 2.

310. A *Commensurable Root* is a real root, which can be exactly expressed by whole numbers or fractions.

311. *Problem. To find the commensurable roots of the equation*

$$x^n + a\,x^{n-1} + b\,x^{n-2} + \&c. + l\,x + m = 0,$$

in which a, b, *&c. are all integers, either positive or negative.*

Solution. Let one of the commensurable roots be, when reduced to its lowest terms,

$$x = \frac{p}{q}.$$

Method of finding Integral Roots.

As this root must verify the given equation, we have

$$\frac{p^n}{q^n} + a\frac{p^{n-1}}{q^{n-1}} + b\frac{p^{n-2}}{q^{n-2}} + \&c. + m = 0;$$

whence, multiplying by q^{n-1}, and transposing, we obtain

$$\frac{p^n}{q} = -a p^{n-1} - b p^{n-2} q - \&c. \ldots - m q^{n-1};$$

and, therefore, as the second member is integral, the first member must also be integral, or we must have

$$q = 1,$$

whence

$$x = p;$$

that is, ***every commensurable root of the given equation must be an integer.***

Again, the substitution of

$$x = p,$$

in the given equation, produces

$$p^n + a p^{n-1} + \&c. \ldots + k p^2 + l p + m = 0;$$

whence, dividing by p, and transposing, we obtain

$$\frac{m}{p} = -l - k p - \&c. \ldots - a p^{n-2} - p^{n-1};$$

and, therefore, as the second member is integral, the first member must be so likewise; that is, ***every integral root must be a divisor of* m.**

If, now, we denote by m',

$$m' = \frac{m}{p} + l,$$

the preceding equation gives, by transposing and dividing by p,

$$\frac{m'}{p} = -k - i p - h p^2 - g p^3 - \&c. - a p^{n-3} - p^{n-2}.$$

so that this ***integral root must likewise be a divisor of* m′**

Method of finding Integral Roots.

In the same way, if we use m'', m''', m^{iv}, &c. as follow

$$m'' = \frac{m'}{p} + k,$$

$$m''' = \frac{m''}{p} + i,$$

$$m^{\text{iv}} = \frac{m'''}{p} + h,$$

&c., &c.;

this integral root must be a divisor of m'', m''', m^{iv}, &c., and the last condition to be satisfied is

$$m^{[n-1]} + p = 0, \text{ or } m^{[n-1]} = -p.$$

Hence to find all the commensurate roots of the given equation, write in the same horizontal line all the integral divisors of m, *which are contained between the extreme limits of the roots.*

Write below these divisors all the corresponding values of m', m'', *&c., which are integral, remembering that a divisor cannot be a root, when the value which it gives for either* m', m'', m''', *&c., is fractional.*

Proceed in this way till the values of $m^{[n-1]}$ *are obtained, and those divisors only are roots which give* $-p$ *for the value of this quantity.*

312. EXAMPLES.

1. Find the commensurable roots of the equation

$$x^5 - 19x^3 + 34x^2 + 12x - 40 = 0.$$

Solution. The extreme limits of the real roots are $+7{\cdot}4$, and $-6{\cdot}9$. Hence we have

m =	-40;							
p =	5,	4,	2,	1,	-1,	-2,	-4,	-5;
m' =	4,	2,	-8,	-28,	52,	32,	22,	20:
m'' =	,	,	30,	6,	-18,	18,	,	30;
m''' =	,	,	-4,	-13,	-1,	-28,	,	-25;
m^{IV} =	,	,	-2,	-13,	1,	14,	,	5;

and, therefore, 2, -1, and -5 are roots of the given equation, and its first member, divided by the factor

$$(x-2)(x+1)(x+5) = x^3+4x^2-7x-10,$$

gives the quotient

$$x^2-4x+4;$$

and, therefore, the remaining roots are those of the equation

$$x^2-4x+4=0,$$

which are equal to each other, and each is

$$x=2.$$

2. Find the commensurable roots of the equation

$$x^8-3x^7-10x^6-2x^4+6x^3+21x^2-3x-10=0.$$

Ans. 5, 1, -1, and -2.

3. Find all the roots of the equation

$$x^4+x^3-24x^2+43x-21=0$$

which has commensurable roots.

Ans. 1, 3, $-\frac{5}{2}\pm\frac{1}{2}\sqrt{53}$.

4. Find all the roots of the equation

$$x^3-6x^2+19x-44=0$$

which has a commensurable root.

Ans. 4, and $1\pm\sqrt{-10}$.

5. Find all the roots of the equation

$$x^4-10x^3+35x^2-50x+24=0$$

which has commensurable roots.

Ans. 1, 2, 3, 4.

Commensurable Roots of any Equation.

6. Find all the roots of the equation

$$x^5 - 3x^4 - 8x^3 + 24x^2 - 9x + 27 = 0$$

which has commensurable and equal roots.

Ans. 3, -3, and $\pm\sqrt{-1}$

7. Find all the roots of the equation

$$x^5 - 23x^4 - 48x^3 + 95x^2 + 400x + 375 = 0$$

which has commensurable and also equal roots.

Ans. 3, 5, and $-2 \pm \sqrt{-1}$.

313. *Problem. To find the commensurable roots of an equation.*

Solution. Reduce the equation to the form

$$A x^n + B x^{n-1} + \&c. \ldots + L x + M = 0,$$

in which A, B, &c., are all integers, either positive or negative.

Substitute for x the value

$$x = \frac{y}{A},$$

and the equation becomes

$$\frac{y^n}{A^{n-1}} + \frac{B y^{n-1}}{A^{n-1}} + \frac{C y^{n-2}}{A^{n-2}} + \&c. \ldots + \frac{L y}{A} + M = 0;$$

which, multiplied by A^{n-1}, is

$$y^n + B y^{n-1} + A C y^{n-2} + \&c. \ldots + A^{n-2} L y + A^{n-1} M = 0.$$

The commensurable roots of this equation may be found, as in the preceding article, and being divided by A, *will give the commensurable roots of the required equation.*

314. *Scholium.* The substitution of

$$x = \frac{y}{A}$$

is not always the one which leads to the most simple result. But when A has two or more equal factors, it is often the case that the substitution

$$x = \frac{y}{A'}$$

leads to an equation of the desired form, A' being the product of the prime factors of A, and each factor need scarcely ever be repeated more than once.

315. EXAMPLES.

1. Find the commensurable roots of the equation

$$64\,x^4 - 328\,x^3 + 574\,x^2 - 393\,x + 90 = 0.$$

Solution. We have, in this case,

$$A = 64 = 2^6;$$

hence we may take A' equal to some power of 2; and it is easily seen that the third power will do, so that we may make

$$x = \tfrac{1}{8}\,y.$$

Hence the given equation becomes

$$y^4 - 41\,y^3 + 574\,y^2 - 3144\,y + 5760 = 0.$$

The commensurable roots of which are found, as in art 311, to be

$$y = 4,\ 6,\ 15,\ \text{and } 16;$$

so that the roots of the given equation are

$$x = \tfrac{1}{2},\ \tfrac{3}{4},\ 1\tfrac{7}{8},\ \text{and } 2.$$

2. Find the commensurable roots of the equation

$$8\,x^3 + 34\,x^2 - 79\,x + 30 = 0.$$

Ans. $\frac{1}{2}$, $\frac{5}{4}$, and -6

3. Find the commensurable roots of the equation

$$24\,x^3 - 26\,x^2 + 9\,x - 1 = 0.$$

Ans. $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

4. Find the commensurable roots of the equation

$$3\,x^3 - 14\,x^2 + 21\,x - 10 = 0.$$

Ans. 1, $\frac{5}{3}$, and 2.

5. Find the commensurable roots of the equation

$$8\,x^4 - 38\,x^3 + 49\,x^2 - 22\,x + 3 = 0.$$

Ans. $\frac{1}{2}$, $\frac{1}{4}$, 1, and 3.

6. Find all the roots of the equation

$$6\,x^3 + 7\,x^2 + 39\,x + 63 = 0$$

which has a commensurable root.

Ans. $-\frac{3}{2}$, and $\frac{1}{6} \pm \frac{1}{6}\sqrt{-251}$.

7. Find the commensurable roots of the equation

$$9\,x^6 + 30\,x^5 + 22\,x^4 + 10\,x^3 + 17\,x^2 - 20\,x + 4 = 0.$$

Ans. $\frac{1}{3}$ and -2.

CHAPTER IX.

CONTINUED FRACTIONS.

316. A *continued fraction* is one whose numerator is unity, and its denominator an integer increased by a fraction, whose numerator is likewise unity, and which may be a continued fraction.

Thus,

$$\cfrac{1}{a+\cfrac{1}{b}} \text{ and } \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\&c.}}}}$$

are continued fractions.

317. *Problem. To find the value of a continued fraction which is composed of a finite number of fractions.*

Solution. Let the given fraction be

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}}$$

Beginning with the last fraction, we have successively

$$c+\frac{1}{d}=\frac{cd+1}{d}$$

$$\frac{1}{c+\frac{1}{d}}=\frac{d}{cd+1}$$

$$b+\cfrac{1}{c+\cfrac{1}{d}} = b + \frac{d}{cd+1} = \frac{b(cd+1)+d}{cd+1}$$

$$\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}} = \frac{cd+1}{b(cd+1)+d} = \frac{cd+1}{(bc+1)d+b}$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}} = \frac{ad(bc+1)+ab+cd+1}{(bc+1)d+b}$$

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}} = \frac{(bc+1)d+b}{ad(bc+1)+ab+cd+1} = \frac{(bc+1)d+b}{(ab+1)cd+ad+ab+1};$$

and this method can easily be applied in any other case.

318. EXAMPLES.

1. Find the value of the continued fraction

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}}$$

Ans. $\frac{13}{30}$.

2. Find the value of the continued fraction

$$\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2}}}$$

Ans. $\frac{7}{30}$.

319. ***Problem.*** *To find the value of an infinite continued fraction.*

Solution. Let the fraction be

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\&c.}}}$$

An approximate value of this fraction is obviously obtained by omitting all its terms beyond any assumed fraction, and obtaining the value of the resulting fraction, as in the previous article.

Thus we obtain, successively,

$$\frac{1}{a} = \frac{1}{a} \quad \text{1st approx. value.}$$

$$\cfrac{1}{a+\cfrac{1}{b}} = \frac{b}{ab+1} \quad \text{2d approx. value.}$$

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}} = \frac{bc+1}{(ab+1)c+a} \quad \text{3d approx. value.}$$

&c., &c.

and each of these values is easily shown to be more accurate than the preceding; for the second value is what the first becomes by substituting, for the denominator a, the more accurate denominator $a+\frac{1}{b}$; the third is what the second becomes by substituting, for the denominator b, the more accurate denominator $b+\frac{1}{c}$; and so on.

320. *Theorem. The numerator of any approximate value, as the* n*th, is obtained from the numerators of the two preceding approximate values, the* $(n-1)$*st, and the* $(n-2)$*nd, by multiplying the* $(n-1)$*st numerator by the* n*th denominator contained in the given continued fraction, and adding to the result the numerator of the* $(n-2)$*nd approximate value.*

The denominator of the n*th approximate value is obtained in the same way from the two preceding denominators.*

Demonstration. Let the $(n-3)$rd, $(n-2)$nd, $(n-1)$st, and nth approximate values be, respectively,

$$\frac{K}{K'}, \frac{L}{L'}, \frac{M}{M'}, \text{ and } \frac{N}{N'},$$

and let the $(n-1)$st and the nth denominators, contained in the given continued fraction, be p and q.

We shall suppose the proposition demonstrated for the $(n-1)$st approximate value, and shall prove that it can thence be continued to the nth value; that is, we shall suppose it proved that

$$\frac{M}{M'} = \frac{pL+K}{pL'+K'}$$

Now it is plain, from the remarks at the end of the preceding article, that the nth value is deduced from the $(n-1)$st, by changing p into $p+\frac{1}{q}$; which change, being made in the preceding value, gives

$$\frac{N}{N'} = \frac{\left(p+\frac{1}{q}\right)L+K}{\left(p+\frac{1}{q}\right)L'+K'} = \frac{(pL+K)q+L}{(pL'+K')q+L'}.$$

Hence we have, by substituting

$$M = p\,L + K,$$
$$M' = p\,L' + K';$$
$$\frac{N}{N'} = \frac{M\,q + L}{M'\,q + L'},$$

that is, the value required to satisfy the theorem.

If, therefore, it can be shown that the proposition is true for any approximate value, it follows that it must be true for every succeeding value. But the comparison of the values given in the preceding article shows that it is true for the third value, and therefore for every succeeding value.

321. *Theorem. If two succeeding approximate values are reduced to a common denominator equal to the product of their denominators, the difference of their numerators is unity.*

Demonstration. Let the $(n-2)$nd, $(n-1)$st, and nth approximate values be

$$\frac{L}{L'},\ \frac{M}{M'},\ \text{and}\ \frac{N}{N'} = \frac{M\,q + L}{M'\,q + L'},$$

the difference between the $(n-2)$nd and $(n-1)$st is

$$\pm\,\frac{L\,M' - L'\,M}{L'\,M'};$$

and that between the $(n-1)$st and nth is

$$\pm\,\frac{M'\,N - M\,N'}{M'\,N'} = \pm\,\frac{(M\,M' - M\,M')\,q + M'\,L - M\,L'}{M'\,N'}$$
$$= \pm\,\frac{L\,M' - L'\,M}{M'\,N'};$$

of both which differences the numerators are the same; and, therefore, this is always the case.

Approximate Value compared with True Value.

Now the first and second approximate values, as given in art. 319, are, when reduced to a common denominator,

$$\frac{ab+1}{a(ab+1)} \text{ and } \frac{ab}{a(ab+1)};$$

the difference of the numerators of which is 1; and, therefore, unity must always be the difference of two such numerators.

322. *Theorem. The approximate values of a continued fraction are alternately larger and smaller than its true value, the first being larger, the second smaller, and so on alternately.*

Demonstration. Since, in the preceding demonstration, the subtraction of the $(n-1)$st value from the $(n-2)$nd, gave a fraction having the same numerator as that obtained, by its subtraction from the nth; we see that if the $(n-1)$st value is larger than the $(n-2)$nd, it must also be larger than the nth; and if the $(n-1)$st is smaller than the $(n-2)$nd, it is also smaller than the nth.

But the true value is, by art. 319, nearer the $(n-1)$st value than the $(n-2)$nd, and nearer the nth than the $(n-1)$st; so that when the $(n-1)$st value is larger than the $(n-2)$nd, the true value must likewise be larger than the $(n-2)$nd, and smaller than the $(n-1)$st, and so on alternately; but when the $(n-1)$st value is smaller than the $(n-2)$nd, the true value must be smaller than the $(n-2)$nd, and larger than the $(n-1)$st, and so on alternately.

Now the first value is, by the preceding article, larger than the second, and therefore the true value is smaller than the first, larger than the second, and so on alternately.

323. *Theorem. Each approximate value of a continued fraction differs from the true value by a*

quantity less than the fraction whose numerator is unity, and whose denominator is the square of the denominator of this approximate value.

Demonstration. Let the denominator of the two successive approximate values be M' and N'; N' must, by art. 320, be larger than M'; and the difference between these two values must be

$$\frac{1}{M'N'}.$$

But, by the preceding article, the true value is contained between these two approximate values, and therefore differs from either of them by a quantity less than their difference.

Now, since

$$M' < N',$$

we have

$$M'^2 < M'N',$$

and

$$\frac{1}{M'^2} > \frac{1}{M'N'};$$

so that the true value must differ from the approximate value, whose denominator is M', by a quantity less than

$$\frac{1}{M'^2},$$

that is, less than a fraction whose numerator is 1, and denominator M'^2.

324. *Problem. To transform any quantity into a continued fraction.*

Solution. Let X be the quantity to be transformed. Find the greatest integer contained in X, and denote it by A, and denote the excess of X above A by the fraction $\frac{1}{x'}$; and we have

Approximate Value compared with True Value.

$$A + \frac{1}{x'} = X,$$

and

$$x' = \frac{1}{X - A}.$$

From this value of x′, *find the greatest integer contained in* x′, *and denote it by* a, *and the excess of* x′ *above* a *by* $\frac{1}{x''}$; whence

$$x'' = \frac{1}{x' - a},$$

from which the greatest integer contained in x″ *is to be found, and so on ;* so that we have

$$X = A + \cfrac{1}{a + \cfrac{1}{a' + \&c.}}$$

325. EXAMPLE.

Transform $\frac{865}{351}$ into a continued fraction.

Solution. We have, in this case, successively,

$$\begin{aligned}
A &= 2; \\
x'' &= \frac{1}{\frac{865}{351} - 2} = \tfrac{351}{263}, \\
a &= 1; \\
x'' &= \tfrac{263}{88}, \\
a' &= 2; \\
x''' &= \tfrac{88}{87}, \\
a'' &= 1; \\
x^{\text{IV}} &= 87 = a''',
\end{aligned}$$

and the required continued fraction is

$$\frac{965}{351} = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{87}}}}.$$

326. *Corollary. The values of* a′, a″, *&c., in the case of a vulgar fraction, are evidently the quotients which would be obtained by the process of finding the greatest common divisor of the numerator and denominator of* x′.

The preceding process might therefore be performed as follows:

$$\begin{array}{l}
263\,|\,351\,|\,1 = a \\
\quad\;\; \underline{263} \\
\quad\;\;\; 88\,|\,263\,|\,2 = a' \\
\qquad\quad\;\; \underline{176} \\
\qquad\quad\;\; 87\,|\,88\,|\,1 = a'' \\
\qquad\qquad\;\; \underline{87} \\
\qquad\qquad\;\;\; 1\,|\,87\,|\,87 = a''' \\
\qquad\qquad\qquad\; \underline{87} \\
\qquad\qquad\qquad\;\; 0 .
\end{array}$$

327. *Corollary. If a fraction or ratio is transformed into a continued fraction by the preceding process, the approximate values of this continued fraction are also approximate values of the given fraction or ratio, which are often of great practical use.*

Thus the approximate values of $\frac{965}{351}$, are

$$2, \; 3, \; \tfrac{8}{3}, \; \tfrac{11}{4};$$

of which the last differs from the true value by only $\frac{1}{1404}$

328. EXAMPLES.

1. Find approximate values of the fraction $\frac{251}{764}$.

Ans. $\frac{1}{3}$, $\frac{22}{67}$, $\frac{23}{70}$, and $\frac{114}{347}$.

2. Find approximate values of the fraction $\frac{907}{18564}$.

Ans. $\frac{1}{20}$, $\frac{2}{41}$, $\frac{15}{307}$, $\frac{77}{1576}$, $\frac{169}{3459}$, and $\frac{246}{5035}$.

3. Find approximate values of the fraction $\frac{3215763}{94218374}$.

Ans. $\frac{1}{29}$, $\frac{3}{88}$, $\frac{7}{205}$, $\frac{10}{293}$, $\frac{87}{2549}$, &c.

4. Find approximate values of the fraction 0·245.

Ans. $\frac{1}{4}$ and $\frac{12}{49}$

5. Find approximate values of the fraction 1·27.

Ans. $\frac{4}{3}$, $\frac{5}{4}$, $\frac{14}{11}$, $\frac{33}{26}$, and $\frac{47}{37}$

6. The lunar month consists of 27·321661 days. Find approximate values for this time.

Ans. 27, $\frac{82}{3}$, $\frac{765}{28}$, $\frac{3907}{143}$, &c. days, which show that the moon revolves about 3 times in 82 days; or with greater accuracy, 28 times in 765 days; and with still more accuracy, 143 times in 3907 days.

7. The sidereal revolution of Mercury is 87·969255 days. Find approximate values for this time.

Ans. 88, $\frac{2815}{32}$, &c.

8. The sidereal revolution of Venus is 224·700817 days. Find approximate values for this time.

Ans. 225, $\frac{674}{3}$, $\frac{1573}{7}$, $\frac{2247}{10}$, $\frac{26290}{117}$, &c.

9. The ratio of the circumference of a circle to its diameter is 3·1415926535. Find approximate values for this ratio.

Ans. 3, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, &c.

329. *Corollary. The process of art.* 324 *may be applied to finding the real roots of an equation, the approximate values of which, obtained by this process, can easily be reduced to decimals.*

330. EXAMPLES.

1. Find the real root of the equation

$$x^3 - 3\,x - 8 = 0.$$

Solution. We have, in this case,

$$A = 2,$$

and if we substitute

$$x = 2 + \frac{1}{x'},$$

in the given equation, we obtain

$$6\,x'^3 - 9\,x'^2 - 6\,x' - 1 = 0,$$

whence we have

$$a = 2;$$

and the substitution of

$$x' = 2 + \frac{1}{x''}$$

gives

$$x''^3 - 30\,x''^2 - 27\,x'' - 6 = 0;$$

whence we have

$$a' = 32,$$

and so on.

The approximate values of x are, therefore,

$$2,\ 2\tfrac{1}{2} = 2{\cdot}5,\ 2\tfrac{32}{65} = 2{\cdot}492,\ \&c.$$

2. Find the real root of the equation

$$x^3 - 12\,x - 28 = 0.$$

Ans. $x = 4{\cdot}30213.$

Approximate Roots of Equation.

3. Find the real root of the equation

$$x^3 - 12x^2 + 57x - 94 = 0.$$

Ans. $x = 3\cdot36216$.

331. *Corollary. If the given equation is a binomial one, as in art.* 223, *we can obtain, by this process, a root of any degree whatever.*

332. EXAMPLES.

1. Extract the square root of 5 by means of continued fractions.

Solution. Representing this root by x, we have

$$x^2 = 5,$$

whence

$$A = 2;$$

and the substitution of

$$x = 2 + \frac{1}{x'}$$

gives

$$x'^2 - 4x' - 1 = 0;$$

whence we have

$$a = 4;$$

and the substitution of

$$x' = 4 + \frac{1}{x''}$$

gives

$$x''^2 - 4x'' - 1,$$

which, being precisely the same with the equation for x', we may conclude that

$$4 = a = a' = a'' = a''' = \&c.$$

and the approximating values are

$$2\tfrac{1}{4},\ 2\tfrac{4}{17},\ 2\tfrac{17}{72},\ 2\tfrac{72}{305},\ \&c.;$$

and the value in decimals is

$$2{\cdot}23606.$$

2. Extract the third root of 46 by means of continued fractions.

Ans. 3·58305.

3. Extract the third root of 35 by means of continued fractions.

Ans. 3·271.

4. Extract the square root of 2 by means of continued fractions.

Ans. 1·4142136.

EXPONENTIAL EQUATIONS

AND

LOGARITHMS.

EXPONENTIAL EQUATIONS

AND

LOGARITHMS.

SECTION I.

EXPONENTIAL EQUATIONS.

1. An *Exponential Equation* is one in which the unknown quantity occurs as an exponent.

2. *Problem. To solve the exponential equation*

$$b^x = m.$$

Solution. This equation is readily solved by means of continued fractions, as explained in Alg. Art. 324.

3. EXAMPLES.

1. Solve the equation

$$3^x = 100.$$

Solution. Since we have

$$3^4 = 81,$$

and

$$3^5 = 243,$$

the greatest integer contained in x must be 4. Substituting then

$$x = 4 + \frac{1}{x'},$$

we have

$$3^{4+\frac{1}{x'}} = 100,$$

or

$$3^4\, 3^{\frac{1}{x'}} = 81 \times 3^{\frac{1}{x'}} = 100;$$

and

$$3^{\frac{1}{x'}} = \tfrac{100}{81};$$

which being raised to the power denoted by x', is

$$3 = \left(\frac{100}{81}\right)^{x'}.$$

By raising $\frac{100}{81}$ to different powers, the greatest integer contained in x' is found to be 5. Substituting then

$$x' = 5 + \frac{1}{x''}$$

we have

$$3 = \left(\frac{100}{81}\right)^{5+\frac{1}{x''}} = \left(\frac{100}{81}\right)^{5} \cdot \left(\frac{100}{81}\right)^{\frac{1}{x''}};$$

or

$$3 = \frac{10000000000}{3486784401} \times \left(\frac{100}{81}\right)^{\frac{1}{x''}}.$$

Hence

$$\left(1{\cdot}0460353203\right)^{x''} = \frac{100}{81},$$

from which the greatest integer contained in x'' is found to be 4; and in the same way we might continue the process

The approximate values of x are, then,

$$4,\ 4\tfrac{1}{5},\ 4\tfrac{4}{21},\ = 4{\cdot}19,\ \&c.$$

Solution of Exponential Equations.

2. Find an approximate value for x, in the equation

$$3^x = 15.$$

Ans. $x = 2{\cdot}46.$

3. Find an approximate value for x, in the equation

$$10^x = 3.$$

Ans. $x = 0{\cdot}477.$

4. Find an approximate value for x, in the equation

$$\left(\tfrac{7}{12}\right)^x = \tfrac{3}{4}.$$

Ans. $x = 0{\cdot}53.$

4. *Corollary.* Whenever the values of b and m are both larger or both smaller than unity, the value of x is positive. But when one of them is larger than unity while the other is smaller, the value of x must be negative; for the positive power of a quantity larger than unity must be larger than unity, and the positive power of a quantity smaller than unity is smaller than unity; whereas the negative power, being the reciprocal of the corresponding positive power, must be greater than unity, when the positive power is less than unity, and the reverse.

Hence to solve the equation

$$b^x = m,$$

in which one of the quantities, b and m, is greater than unity, while the other is smaller than unity make

$$x = -y,$$

which gives

$$b^{-y} = m,$$

or

$$\left(\frac{1}{b}\right)^y = m,$$

which may be solved as in the preceding article.

5. EXAMPLES.

1. Solve by approximation the equation

$$5^x = \tfrac{2}{3}.$$

Ans. $x = -0{\cdot}25$

2. Solve by approximation the equation

$$2^x = \tfrac{1}{3}.$$

Ans. $x = -1{\cdot}58.$

SECTION II.

NATURE AND PROPERTIES OF LOGARITHMS.

6. The root of the equation

$$b^x = m$$

is called the *logarithm* of m; and since, by the preceding section, this root can be found for any value which m may have, it follows that every number has a logarithm. The logarithm of a number is usually denoted by *log.* before it, or simply by the letter l.

7. But the value of the logarithm varies with the value of b, and therefore the value of b, which is called the *base of the system* of logarithms, is of great importance; and *the logarithm of a number* may be defined as *the exponent of the power to which the base of the system must be raised in order to produce this number.*

8. *Corollary.* When the base is less than unity, it follows, from art. 3, that the logarithms of all numbers greater than unity are negative, while those of all numbers less than unity are positive.

But when, as is almost always the case, *the base is greater than unity, the logarithms of all numbers greater than unity are positive, while those of all numbers less than unity are negative.*

9. *Corollary.* Since

$$b^0 = 1,$$

it follows, that *the logarithm of unity is zero in all systems.*

10. *Theorem. The sums of the logarithms of several numbers is the logarithm of their continued product.*

Proof. Let the numbers be m, m', m'', &c., and let b be the base of the system; we have then

$$b^{\log. m} = m,$$
$$b^{\log. m'} = m',$$
$$b^{\log. m''} = m'', \text{ \&c.};$$

the product of which is, by art. 28,

$$b^{\log. m + \log. m' + \log. m'' + \&c.} = m\, m'\, m'' \text{ \&c.}$$

Hence, by art. 7,

$$\log. m\, m'\, m'' \text{ \&c.} = \log. m + \log. m' + \log. m'' + \text{\&c.}$$

11. *Corollary.* If the number of the factors, m, m', &c. is n, and if they are all equal to each other, we have

$$\log. m\, m\, m \text{ \&c.} = \log. m + \log. m + \log. m + \text{\&c.}$$

or

$$\log. m^n = n \log. m;$$

that is, *the logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

12. *Corollary.* If we substitute

$$p = m^n,$$

or

$$m = \sqrt[n]{p},$$

in the above equation, it becomes

$$\log. p = n \log. \sqrt[n]{p},$$

or

$$\log. \sqrt[n]{p} = \frac{\log. p}{n};$$

that is, *the logarithm of any root of a number is equal to the logarithm of the number divided by the exponent of the root.*

13. *Corollary.* The equation

$$\log. m\, m' = \log. m + \log. m',$$

gives

$$\log. m' = \log. m\, m' - \log. m;$$

that is, the logarithm of one factor of a product is equal to the logarithm of the product diminished by the logarithm of the other factor; or, in other words,

The logarithm of the quotient is equal to the logarithm of the dividend, diminished by the logarithm of the divisor.

14. *Corollary.* We have, by arts. 13 and 9,

$$\log. \frac{1}{n} = \log. 1 - \log. n$$
$$= -\log. n;$$

that is, *the logarithm of the reciprocal of a number is the negative of the logarithm of the number.*

15. *Corollary.* Since zero is the reciprocal of infinity, we have

$$\log. 0 = - \log. \infty = - \infty;$$

that is, *the logarithm of zero is negative infinity.*

16. *Corollary.* Since we have

$$b^1 = b,$$

the *logarithm of the base of a system is unity.*

17. *Theorem. If the logarithms of all numbers are calculated in a given system, they can be obtained for any other system by dividing the given logarithms by the logarithm of the base of the required system taken in the given system.*

Demonstration. Let b be the base of the given system, and b' that of the required system; and denote by *log.* the logarithms in the given system, and by *log.'* the logarithms in the required system. Taking, then, any number m, we have, by art. 7,

$$b^{\log. m} = m,$$

and

$$b'^{\log.' m} = m;$$

whence

$$b'^{\log.' m} = b^{\log. m}.$$

If we take the logarithms of each member of this equation in the given system, we have, by arts. 11 and 16,

$$\log.' m \times \log. b' = \log m \times \log. b = \log. m,$$

or, dividing by log. b',

$$\log.' m = \frac{\log. m}{\log. b'}.$$

SECTION III.

COMMON LOGARITHMS AND THEIR USES.

18. The base of the system of logarithms in common use is 10.

19. *Corollary.* Hence in common logarithms, we have by arts. 16 and 9,

$$\begin{aligned}
\log.\ 1 &= 0, \\
\log.\ 10 &= 1, \\
\log.\ 100 &= \log.\ 10^2 = 2, \\
\log.\ 1000 &= \log.\ 10^3 = 3, \\
\log.\ 10000 &= \log.\ 10^4 = 4, \\
&\text{\&c., \&c.,}
\end{aligned}$$

also,

$$\begin{aligned}
\log.\ 0{\cdot}1 &= \log.\ 10^{-1} = -1, \\
\log.\ 0{\cdot}01 &= \log.\ 10^{-2} = -2, \\
\log.\ 0{\cdot}001 &= \log.\ 10^{-3} = -3, \\
&\text{\&c., \&c.;}
\end{aligned}$$

that is, *the logarithm of a number, which is composed of a figure* 1 *and cyphers, is equal to the number of places by which the figure* 1 *is removed from the place of units; the logarithm being positive when the figure* 1 *is to the left of the units' place, and negative when it is to the right of the units' place.*

20. *Corollary.* If, therefore, a number is

between 1 and 10, its log. is between 0 and 1,
if between 10 and 100, its log. is between 1 and 2,
if between 100 and 1000, its log. is between 2 and 3,

and so on.

But if between 0·1 and 1, its log. is between —1 and 0,
if between 0·01 and 0·1, its log. is between —2 and —1,
and so on.

Hence, if the greatest integer contained in a logarithm is called its *characteristic*, *the characteristic of the logarithm of a number is equal to the number of places by which its first significant figure on the left is removed from the units' place, the characteristic being positive when this figure is to the left of the units' place, negative when it is to the right of the units' place, and zero when it is in the units' place.*

21. Logarithms have been found of such great practical use, that much labor has been devoted to the calculation and correction of logarithmic tables. In the common tables they are given to 5, 6, or 7 places of decimals. In almost all cases, however, 5 places of decimals are sufficiently accurate; and it is, therefore advisable to save unnecessary labor, and avoid an increased liability to error, by omitting the places which may be given beyond the first five.

22. *Problem. To find the logarithm of a given number from the tables.*

Solution. First. Find the characteristic by the rule of art. 20.

The characteristic is the most important part of the logarithm, and yet the unskilful are very apt to err in regard to

Finding a Logarithm.

it, not appearing to consider that an error of a single unit in its value will give a result 10 times as great or as small as it should be.

If the characteristic thus found is negative, the negative sign is usually placed above it, that this sign may not be referred to the decimal part of the logarithm, which is always positive. But calculators are in the habit of avoiding the perplexity of a negative characteristic by subtracting its absolute value from 10, *and writing the difference in its stead; and, in the use of a logarithm so written, it must not be forgotten that it exceeds the true value by* 10.

Secondly. In finding the decimal part of the logarithm, the decimal point of the given number is to be wholly disregarded, and any cyphers which may precede its first significant figure on the left, or follow its last significant figure on the right are to be omitted.

When the number thus simplified is contained within the limits of the tables, which we shall regard as extending to numbers consisting of four places, the decimal part of its logarithm is found in a horizontal line with its three first figures, and in the column below its fourth figure; the second, third, and fourth figures, when wanting, being supposed to be cyphers.

When the number consists of more than four places, and is therefore, beyond the limits of the tables, point off its first four places on the left and

consider them as integers, regarding the other places as decimals.

Care must be taken not to confound the decimal point thus introduced with the actual decimal point of the number, of which it is altogether independent.

Find, in the tables, the decimal logarithm corresponding to the integral part of the number thus pointed off; and also the difference between this logarithm and the one next above it, that is, the logarithm of the number which exceeds this integral part by unity; this difference is often given in the margin of the tables.

Multiply this difference by the decimal part of the number as last pointed off, and omit in the product as many places to the right as there are places in this decimal part of the number.

The product, thus reduced, being added to the decimal logarithm of the integral part of the number, is the decimal part of the required logarithm.

23. *Corollary.* This process for finding the decimal part of the logarithm of a number, which exceeds the limits of the tables, is founded on the following law, easily deduced from the inspection of the tables.

If several numbers are nearly equal, their differences are proportional to the differences of their logarithms.

24. EXAMPLES.

1 Find the logarithm of 0·00325787.

Solution. The characteristic is -3, instead of which may be written $10 - 3 = 7$.

For the decimal part, the number is to be written

$$3257 \cdot 87;$$

and we have

log. 3258 — log. 3257	=	13
now, multiplying by		·87
and omitting two places		91
on the right,		104
we have		11
which, added to log. 3257	=	51282
gives		51293;

and the required logarithm is

$$\log. 0 \cdot 003257 87 = \bar{3} \cdot 51293,$$

or, it may be written,

$$7 \cdot 51293.$$

2. Find the logarithm of 1·8924. *Ans.* 0·27701.

3. Find the logarithm of 757·823000. *Ans.* 8·87956.

4. Find the logarithm of 0·00041359.

Ans. $\bar{4} \cdot 61657$, or 6·61657

5. Find the logarithm of 0·12345.

Ans. $\bar{1} \cdot 09149$, or 9·09149.

6. Find the logarithm of 99998. *Ans.* 4·99999.

25. *Problem. To find the number corresponding to a given logarithm.*

Solution. First. In finding the figures of the required number, the characteristic is to be neglected.

When the decimal part of the given logarithm is exactly contained in the tables, its corresponding number can be immediately found by inspection.

But when the given logarithm is not exactly contained in the tables, the number, corresponding to the logarithms of the table which is next below it, gives the four first places on the left of the required number.

One or two more places are found by annexing one or two cyphers to the difference between the given logarithm and the logarithm of the tables next below it, and dividing by the difference between the logarithm of the tables next below and that next above the given logarithm.

When tables are used in which the logarithms are given to five places, the accuracy of the corresponding numbers is never to be relied upon to more than 6 places, and rarely to more than 5 places; so that in finding the last quotient, one place is usually sufficient.

Secondly. The position of the decimal point of the required number depends altogether upon the characteristic of the given logarithm, and is easily ascertained by the rule of art. 20; *cyphers being prefixed or annexed when required.*

26. EXAMPLES.

1. Find the number, whose logarithm is 8·19325.

Solution. We have for the logarithm of the tables next below the given logarithm

$$\cdot 19312 = \log.\ 1560.$$

Hence

the diff. between given log. and log. 1560 = 13,
also log. 1561 — log. 1560 = 28,

and the quotient

$$\tfrac{1300}{28} = 46$$

gives the two additional places; so that the six places of the required number are

156046;

and the number is, therefore,

156046000.

2. Find the number, whose logarithm is 2·13511.

Ans. 136·493.

3. Find the number, whose logarithm is 1·76888.

Ans. 58·7328.

4. Find the number, whose logarithm is 0·11111.

Ans. 1·29153.

5. Find the number, whose logarithm is $\bar{2}$·98357.

Ans. 0·0962875.

6. Find the number, whose logarithm, when written 10 more than it should be, is 9·35846.

Ans. 0·22828.

27. ***Problem.*** *To find the product of two or more factors by means of logarithms.*

Solution. Find the sum of the logarithms of the factors, and the number, of which this sum is the logarithm, is, by art. 10, *the required product.*

When the logarithm of any of the factors is written, as in art. 22, 10 *more than its true value, as many times* 10 *should be subtracted from the result as there are such logarithms.*

Involution by Logarithms.

28. EXAMPLES.

1. Find the continued product of 78·052, 0·6178, 341000, 100·008, and 0·0009.

Solution. We find, from the tables,

$$\begin{array}{rl} \text{log. } 78{\cdot}052 = & 1{\cdot}89238 \\ 10 + \text{log. } 0{\cdot}6178 = & 9{\cdot}79085 \\ \text{log. } 341000 = & 5{\cdot}53275 \\ \text{log. } 100{\cdot}008 = & 2{\cdot}00003 \\ 10 + \text{log. } 0{\cdot}0009 = & \underline{6{\cdot}95424} \\ \text{log. } 1479960 \quad & 6{\cdot}17025 \end{array}$$

and the required product is

1479960.

In the sum of the preceding logarithms 20 was neglected, because two of the logarithms were written 10 more than they should be.

2. Find the continued product of 0·0001, 7,9004, 0·56, 0·032569, and 17899·1.

Ans. 0·257792.

3. Find the continued product of 3·1416, 0·559, and 64·01.

Ans. 112·41.

4. Find the continued product of 3·26, 0·0025, 0·25, and 0·003.

Ans. 0·00000611257.

29. *Problem. To find any power of a given number by means of logarithms.*

Solution. Multiply the logarithm of the given number by the exponent of the required power, and

the number, of which this product is the logarithm, is, by art. 11, *the required power.*

When the logarithm of the given number is written 10 *more than it should be, as many times* 10 *must be deducted from the product as there are units in the given exponent.*

30. EXAMPLES.

1. Find the 4th power of 0·98573.

Solution. We have, by the tables,

$$\begin{array}{lr} 10 + \log.\ 0{\cdot}98573 = & 9{\cdot}99375 \\ \text{multiply by} & 4 \\ \hline 10 + \log.\ 0{\cdot}94406 = & 9{\cdot}97500 \end{array}$$

and the required power is

0·94406.

In the above product, 40 should have been neglected, but in order to avoid a negative characteristic, only 30 was neglected, leaving the exponent 10 too large.

2. Find the 3d power of 0·25. *Ans.* 0·015625.

3. Find the 7th power of 3·1416. *Ans.* 3020·28.

4. Find the square of 0 0031422.
Ans. 0·00000987325.

31. *Problem. To find any root of a given number by means of logarithms.*

Solution. Divide the logarithm of the given number by the exponent of the required root, and the number, of which this quotient is the logarithm, is, by art. 12, *the required root.*

When the logarithm of the given number has a negative characteristic, instead of being increased by 10, *it should be increased by as many times* 10 *as there are units in the exponent of the root, and the quotient will in this case exceed its true value by* 10.

32. EXAMPLES.

1. Find the fifth root of 0·028145.

Solution. We have, by the tables,

$$50 + \log.\ 0{\cdot}028145 = 48{\cdot}44940,$$

which, divided by 5, gives

$$10 + \log.\ 0{\cdot}48964 = 9{\cdot}68988,$$

and the required root is

$$0{\cdot}48964.$$

2. Find the cube root of 0·002197. *Ans.* 0·13.

3. Find the 10th root of 0·000000001. *Ans.* 0·12589.

4. Find the square root of 238·149. *Ans.* 15·4317.

33. The *arithmetical complement* of a logarithm is the remainder after subtracting it from 10.

34. *Corollary. The arithmetical complement of the logarithm of a number is, by art.* 14, *and the preceding article, the logarithm of its reciprocal increased by* 10.

35. *Corollary. The most convenient method of finding the arithmetical complement of a logarithm is to subtract the first significant figure on the right from* 10, *and each figure to the left of this figure from* 9.

Arithmetical Complement.

36. EXAMPLES.

1. Find the arithmetical complement of 9·62595.

Ans. 0·37405.

2. Find the arithmetical complement of the logarithm of 6. *Ans*. 9·22185.

3. Find the arithmetical complement of the logarithm of 0·07. *Ans*. 11·15490.

4. Find the reciprocal of 0·01115.

Solution. We have, by the tables,

log. 0·01115	(ar. co.)	11·95273
subtract		10·
log. 89·686		1·95273

and the required reciprocal is

89·686.

5. Find the reciprocal of 2330. *Ans*. 0·00042918.

6. Find the reciprocal of 68·99. *Ans*. 0·014494.

37. *Problem. To find the quotient of one number divided by another by means of logarithms.*

Solution. Subtract the logarithm of the divisor from that of the dividend, and the number, of which the remainder is the logarithm, is, by art. 13, *the required quotient.*

Or, since, by art. 81, multiplying by the reciprocal of a number is the same as dividing by it, *add the logarithm of the dividend to the arithmetical complement of the logarithm of this divisor, and the sum diminished by* 10 *is the logarithm of the quotient.*

Division by Logarithms.

When the logarithm of the dividend is written 10 *more than its true value,* 20 *must be subtracted from the sum, instead of* 10.

38. EXAMPLES.

1. Divide 0·01478 by 0·9243.

Solution. We have, by the tables,

10 + log. 0·01478	8·16967
log. 0·9243 (ar. co.)	10·03419
10 + log. 0·01599	8·20386

and the required quotient is

0·01599.

2. Divide 0·00815 by 0·0025. *Ans.* 3·26.

3. Divide 40·32 by 2240. *Ans.* 0·018.

4. Divide 0·875 by 25. *Ans.* 0·035.

5. Divide 0·013 by 0·13. *Ans.* 0·1.

39. *Corollary. The value of any fraction may be found by adding together the logarithms of all the factors of the numerator and the arithmetical complements of the logarithms of all the factors of the denominator, and subtracting from the sum as many times* 10 *as there are arithmetical complements plus as many times* 10 *as there are logarithms of the factors of the numerator, which are written greater than their true value by* 10; *the remainder is the logarithm of the fraction.*

40. EXAMPLES.

1 Find the value of the fraction

$$\frac{(0\cdot327)^7 \times \sqrt{19\cdot81}}{(1\cdot23)^{\frac{1}{3}} \times (0\cdot005)^2}.$$

Solution. We have, from the tables,

$10 +$ log. $(0\cdot327)^7$		6·60185
log. $\sqrt{19\cdot81}$		0·64844
log. $(1\cdot23)^{\frac{1}{3}}$	(ar. co.)	9·97003
log. $(0\cdot005)^2$	(ar. co.)	14·60206
log. 66·433		1·82238

and the required value is

66·433.

2. Find the value of the fraction

$$\sqrt[5]{\left(\frac{0\cdot365 \times \sqrt{2}}{788}\right)}.$$

Ans. 0·2308.

3. Find the value of the fraction

$$\sqrt[9]{\left(\frac{347 \times \sqrt[7]{0\cdot0073}}{126 \times \sqrt[3]{\frac{4}{5}}}\right)}.$$

Ans. 1·0666.

41. *Corollary. The logarithm of the fourth term of a proportion is found by adding together the arithmetical complement of the logarithms of the first term and the logarithms of the second and third terms.*

42. EXAMPLES.

1. Find the fourth term of the proportion

$$963 : 1279 = 8{\cdot}7 : x.$$

Solution.

log. 963 (ar. co.)	7·01637
log. 1279	3·10687
log. 8·7	0·93952
log. 11·555	1·06276

and we have

$$x = 11{\cdot}555.$$

2. Find the fourth term of the proportion

$$0{\cdot}0138 : 0{\cdot}319 = 76{\cdot}5 : x.$$

Ans. $x = 1768{\cdot}3$.

43. *Problem. To solve the exponential equation*

$$a^x = m,$$

by means of logarithms.

Solution. The logarithms of the two members of this equation give

$$x \text{ log. } a = \text{log. } m;$$

hence

$$x = \frac{\text{log. } m}{\text{log. } a},$$

or

$$\text{log. } x = \text{log. log. } m - \text{log. log. } a;$$

that is, *the root of this equation is equal to the logarithm of* m *divided by the logarithm of* a, *and this quotient may be obtained by the aid of logarithms.*

44. EXAMPLES.

1. Solve the equation

$$625^x = 3125.$$

Solution. We have, from the tables,

$$\log.\ 3125 = 3{\cdot}49485,$$
$$\log.\ \ 625 = 2{\cdot}79588;$$

and also

$$\log.\ \log.\ 3125 = \log.\ 3{\cdot}49485 = 0{\cdot}54343$$
$$\log.\ \log.\ \ 625 = \log.\ 2{\cdot}79588 = \underline{0{\cdot}44652}$$
$$\log.\ x = \log.\ 1{\cdot}25 \qquad 0{\cdot}09691;$$

hence

$$x = 1{\cdot}25.$$

2. Solve the equation

$$3^x = 15.$$

Ans. $x = 2{\cdot}464.$

3. Solve the equation

$$10^x = 3.$$

Ans. $x = 0{\cdot}477$

THE END.